Student sensemaking of proofs at various distances: The role of epistemic, rhetorical, and ontological distance in the peer review process

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This manuscript focuses on how students make sense of proofs. Participants were students who engaged in peer-review conferences of each other's attempted proofs in a graduate-level real analysis course for mathematics teachers. Building on the concept of distance from conversational analysis, we distinguish how three types of distance (epistemic, rhetorical, and ontological) between a student and a particular claim influence sensemaking. This article also explores the impact of students' sensemaking on their perceptions of proof.

Keywords: Distance; Peer assessment; Peer review; Proof; Proof comprehension; Student perceptions

Introduction

A wealth of research describes the teaching and learning of proof (e.g., Byrne, 2014; Moore, 1994). This article extends existing work to highlight social factors related to learning proof. Beyond proof construction, students can engage in a variety of proof sensemaking practices (Selden & Selden, 2017). By engaging with these practices, students experience the tentative nature of mathematics in the making, challenging the notion that mathematics comes pre-packaged in a neatly finished form (Hersh, 1997).

Our study focused on 13 students who engaged in peer-review conferences of each other's proofs in a graduate-level real analysis course for mathematics teachers. To understand student sensemaking, we explored how the relationship between a student and a particular claim (formally defined below as distance) impacted sensemaking. Our study also explored the resulting impact on student perceptions of proof. We important theoretical contributions to the study of proof, by proposing a new construct—*ontological distance*—and by empirically documenting the role of various forms of distance in proof sensemaking.

Theoretical Framing

Proof as a Social Practice

From a situated perspective, a proof is a logical argument that meets negotiated social standards (Weber, 2008). We use *proof* to denote an argument that is deemed complete and correct, contrasting an *attempted proof*, which may not meet the negotiated standards (Byrne, 2014). Beyond constructing proofs, mathematical proficiency requires students to make sense of proofs. In this article, we define *sensemaking* as the set of practices typically associated with proof comprehension and validation (Selden & Selden, 2017). Validation focuses on determining the correctness of an attempted proof (Weber, 2008), whereas comprehension describes how one understands a proof (Mejia-Ramos et al., 2012). Below in our methods, we operationalize sensemaking through four constructs (meaning, logic, justification, and holistic) from a model for the assessment of proof comprehension (Mejia-Ramos et al., 2012).

Sensemaking and Distance

Conversational analysis provides tools for understanding the relationship between students and particular claims made in attempted proofs. Here we draw upon prior work that distinguished two types of *distance* between individuals and claims: *rhetorical* and *epistemic* (Conlin & Scherr, 2018). In addition, we propose a third type of distance: *ontological*. Our analyses below explicate how various forms of distance influenced sensemaking. These types of distance are distinct, but interrelated.

Rhetorical distance refers to the relationship between an individual and the author of a claim (Conlin & Scherr, 2018). An author can be an external *authority*, a *peer,* or *oneself.* Proofs from an *authority* are typically assumed to be correct. Although such proofs provide a model of expert practice, they may use formalisms that are unfamiliar to students (Selden & Selden, 2014). In contrast, a *peer's* attempted proof is typically communicated less formally, but it cannot be assumed to be correct. Similarly, the veracity of attempted proofs constructed by *oneself* is unknown, and the proximity of students to their own work inhibits their ability to see any flaws (Reinholz, 2016). The *rhetorical distance* between a student and a claim impacts how a student is likely to interpret it. For instance, students are more likely to assume a proof from an authority is correct without needing to validate it, as compared to a proof from a peer.

Epistemic distance refers to what extent an individual endorses a claim (Conlin & Scherr, 2018). Epistemic distance is fluid and may evolve throughout a conversation. For instance, individuals may endorse claims, or they may intentionally distance themselves (e.g., by hedging, joking, quoting, changing tone of voice) as a mechanism to engage tentative understandings more safely. As students create distance, they become dissociated from particular claims, and thus, they can "save face" if a claim turns out to be incorrect (Conlin & Scherr, 2018).

In this article we propose the additional concept of *ontological distance,* which refers to the relationship between an individual and the medium through which a claim is represented. For instance, an individual can represent a claim through spoken word (in the moment) or by embodying it into a particular artifact (e.g., inscriptions on paper). Externalizing claims makes them easier to engage with as distinct entities. In addition, the nature of a medium influences how individuals may engage with the idea embodied in it. For instance, scratch-work on a whiteboard conveys a sense of tentativeness that differs from a solution typeset in Latex. Externalizing claims also adds temporality, insofar that one can compare past claims and present claims. Thus, *ontological distance* can help separate individuals from claims to support more critical engagement.

Perceptions of Proof

Hersh (1997) contrasts two types of mathematics: the "front door" and the "back door." Front-door mathematics comes in a neatly packaged form (e.g., textbooks). Back-door mathematics is more informal, messy, and often incomplete. Back-door mathematics precedes front-door mathematics that is later disseminated publicly. A complete picture of mathematics requires familiarity with both the front and back doors (Hersh, 1997).

When students *only* engage with front-door mathematics, it often creates a false perception that mathematics is about *memorization*, potentially leading to feelings of isolation and *alienation* (Ernest, 1992). In contrast, when students experience mathematics through both doors, it supports *exploration* and *connection* (Boaler & Greeno, 2000). As we argue below, engaging in the sensemaking of proofs at various distances allows students to coordinate both the formal and informal, leading to a deeper appreciation of mathematics.

Distance in the Peer Review of Proofs

Peer review involves students exchanging feedback on their attempted proofs through a peer-review conference, revising, and submitting revised proofs (Byrne, 2014). To date, this common pedagogical practice remains undertheorized, and has not drawn heavily from the rich literature on self- and peer-assessment (e.g., Reinholz, 2016; Sadler, 1989). In this paper, we posit a relationship between sensemaking and perceptions that develops as students engage with attempted proofs through peer-review conferences. This engagement is mediated by various forms of distance, which provide a productive learning environment. Peer-review conferences provide an abundance of attempted proofs, allowing students to coordinate ideas from various *rhetorical distances* (i.e. authorities, peers, and themselves). Through exposure to attempted proofs at different gradations of quality, students are supported to develop a more robust conception of a high-quality proof (Sadler, 1989).

Students construct their attempted proofs before a conference takes place, and throughout the conference they gradually refine their ideas. The draft creates *ontological distance* between a student and their initial thinking, which supports revision of their ideas. Students can also create *epistemic distance* throughout the free-flowing conversation, which makes it safer for them to engage productively with their emergent understandings. Each of these forms of distance supports students to engage with both front-door and back-door mathematics, enriching their perceptions of proof. In turn, as student perceptions shift, it will further influence how they engage with sensemaking practices (e.g., students who perceive mathematics as a messy, collaborative discipline will be more likely to take risks).

This study explored the role of distance in peer-review conferences through two research questions: 1) what role did various forms of distance play in students' sensemaking practices? And 2) how were these forms of distance and sensemaking practices related to students' perceptions of proof?

Methods

Context and Participants

The study took place in *Graduate Analysis for Teachers*, offered at a large, PhD-granting Hispanic-Serving Institution in the US. The course serves students earning a Mathematics Education Master's Degree. These students complete a variety of higher-level mathematics and education-focused courses. Some courses are designed specifically for teachers to engage with higher-level mathematics content (like the course in this study), while others are standard graduate-level mathematics courses. All 13 students (8 women, 5 men) enrolled in the course had a bachelor's degree in mathematics and consented to the study, which was completed with approval from an Institutional Review Board.

Course Content

The course met for 75 minutes twice weekly. It surveyed graduate-level real analysis content (e.g., sets, metric spaces, curves, series, etc.). Below we focus on continuity and uniform continuity. The course coverage of these topics included the definitions of these concepts on real numbers and over abstract metric spaces and how these concepts related to sequences of functions, convergence, and uniform convergence. There was no assigned text, but the instructor (the first author) drew from a variety of sources (e.g., Bressoud, 2007; Rudin, 1964). Because many students had taken undergraduate real analysis over five years ago, concepts like convergence, converse/contrapositive, and proof techniques were also revisited.

Instructional Design

There were three primary instructional methods: inquiry explorations, a special form of two-column proofs, and peer review.

Inquiry explorations. Group work was used to explore mathematics. A typical lesson involved a 1) short introduction, 2) student exploration in groups, and a 3) plenary discussion. For example, an inquiry exploration would require students to construct the Lebesgue curve as a sequence of functions and map quadrants on the square to ternary expansions. After group work time, the students would present their ideas and the instructor would use discourse moves to synthesize their work (Reinholz, 2018).

A special type of two-column proofs. All proofs were structured using a two-column format, adapted from the familiar format used in many secondary school geometry courses. The left column consisted of formal mathematics and the right column was for annotations. Annotations allowed a proof's author to comment on their thought process, include relevant diagrams and definitions, and indicate areas of confusion. Annotations were also used to provide feedback for peer review. Authors used pencil for their annotations, while reviewers used pen. This design was a scaffold intended to help students codify their intuitive ideas into a formal proof (Mamona-Downs & Downs, 2010).

Peer review. Students completed weekly homework assignments, with one problem designated for peer review. The peer review process, called Peer-Assisted Reflection (PAR; Reinholz, 2016) had students: 1) create a draft, 2) exchange peer feedback, 3) revise their drafts, and 4) turn in a final submission. Students annotated their drafts and submitted solutions using the two-column format. Submissions also included a response to reviewer comments, loosely modeled on professional academic peer review (e.g., in Mathematics Education). Steps 1), 3), and 4) of peer review occurred outside of class, while step 2), exchanging peer feedback, occurred during class. Students were assigned random partners (with one group of three). Students had five minutes for silent reading and annotating each other's attempted proofs and approximately fifteen minutes for discussion. Students then had two days to revise their work and turn the final draft in the next class session.

Data Sources

We collected two data corpuses for analysis.

Student sensemaking. Students audio recorded their peer conferences for extra credit (N = 20 conferences recorded). Conferences were only sampled from the second half of the semester, after students were already familiar with the process. Student written work (draft, feedback, and solution) was also collected. Students who did not wish to participate could receive extra credit by completing additional homework problems, but no students opted to do so.

Perceptions of proof. The final exam included a five-question take-home essay: 1) How does your experience with proof in this class compare to your experiences with proof in other classes? 2) What does a "good proof" look like? What information does it provide for a reader?

3) What should you include in an annotation of your own proof, to make it easier for a peer to understand? 4) What type of feedback should you provide to a peer (when annotating their work) to help them create a good proof? 5) What are the implications for the teaching and learning of proof?

Analysis

Student sensemaking. Drawing on a proof comprehension framework (Mejia-Ramos et al., 2012), we developed a coding framework (see Table 1). We chose a unit of analysis as an entire peer conference, because our goal was to identify the presence of the four codes in Table 1 and explore their relationship to types of distance. Because the codes were present throughout segments a conversation, rather than occurring as discrete events, the total quantity of instances was of less concern. The first author coded all 20 recorded peer conferences, and the second author double-coded four conferences (20% of the dataset). A single disagreement was resolved through discussion.

Table 1. Sensemaking codes.

Code	Evidence	Example
Meaning	Students focus on the meaning of words and statements: exploring definitions, revoicing statements, or giving examples of specific terms.	"We are wondering what the difference between the definitions of continuity and uniform continuity are."
Logic	Students focus on logical connections: identifying the purpose of sentences, identifying the overall proof framework (e.g., contradiction, contrapositive, induction, cases), or changing logical statements (e.g., through negation).	“choose a point where it's not uniformly continuous and then use the negation”
Justification	Students connect claims to data and warrants: making implicit warrants explicit, by identifying data supporting a claim, or identifying claims supported by a given statement.	“The only thing that I don't know is how we can conclude this.”
Holistic	Students describe proof as a whole: summarizing main ideas, methods, or connecting techniques between proofs.	"It goes back to the ball idea (creating a neighborhood)."

Perceptions of proof. Drawing from the learning ecologies framework (Boaler & Greeno, 2000), we developed four codes (connection, exploration, alienation, and memorization) through a four-step process (see Table 2). Each student essay was reviewed, and memos summarized how connection, exploration, alienation, and memorization were evident. Tentative thematic codes (Corbin & Strauss, 2007) were developed to capture the focus of student perceptions. Third, all 13 essays were analyzed with these four themes with specific text highlighted that referred to student perceptions of proof. In the fourth and final step, individual student summaries were written. The second author used the developed coding scheme to double-code four of the essays (30% of the dataset). Each essay could receive a total of eight codes, corresponding to four constructs for each of past and prior experiences (and there were 4 essays, for a total of 32 coded units). The single disagreement was resolved through discussion (for an agreement rate of 31/32).

Table 2. Perceptions codes.

Code	Evidence	Example
Connection	Student describes feelings of ownership, personal connection, agency, or control. Student describes collaboration or peer support.	"Proof has begun to feel personal"
Exploration	Student describes practices of proving: generating ideas, articulating reasoning, or revision. Student describes figuring something out, not just memorizing.	"I can figure it out on my own rather than look it up online."
Alienation	Student describes negative emotions (e.g., frustration, confusion, trauma) or the inaccessibility of mathematics (e.g., isolation, exclusion, or elitism).	"I also had a traumatic experience"
Memorization	Students describes memorizing or copying from the Internet as a learning or teaching strategy.	"All you have to do is memorize all these proofs"

Distance. After coding student transcripts according to sensemaking and perceptions, we completed another round of analysis that focused on distance. Here, we identified portions of the transcript of interest, and categorized them according to type of distance. All categorizations and inferences were discussed together by the author team.

Results: Student Sensemaking

The results of this section answer the research question: *What role did various forms of distance play in students' sensemaking practices?* To answer this question, we begin with two extended episodes from student conferences that highlight multiple types of sensemaking (in **bold**) and how they related to multiple types of distance (in *italics*). For ease of interpretation, both episodes come from the same homework problem, which was focused on uniform continuity.[1] In particular, students were asked to:

1) informally describe the difference between continuity and uniform continuity;
2) find all intervals on which $f(x) = x^2$ (defined on R) is continuous;
3) find all intervals on which $f(x) = x^2$ (defined on R) is uniformly continuous;

and prove their results using the definitions of continuity, uniform continuity, and their negations. Only a single student provided a complete and correct proof that $f(x) = x^2$ is uniformly continuous on any closed and bounded finite interval (and not uniformly continuous on R).

Episode 1: Negotiating tentative understandings and an external authority

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[1] **Continuity**: Let $f: D \rightarrow R$, where D is a subset of R. We say that f is *continuous* on D, if for every x_0 *in* D, f is continuous at x_0. We say f is continuous at x_0 if, for every $\varepsilon > 0$ there exists a $\boldsymbol{\delta} > 0$ such that for all x in D with $|x - x_0| < \boldsymbol{\delta}$, $|f(x) - f(x_0)| < \varepsilon$.
Uniform continuity: Let $f: D \rightarrow R$, where D is a subset of R. We say that f is *uniformly continuous* on D, if for every $\varepsilon > 0$ there exists a $\boldsymbol{\delta} > 0$ such that for all s,t in D with $|s-t| < \boldsymbol{\delta}$, $|f(s) - f(t)| < \varepsilon$.

The episode focused primarily on parts 2 and 3 of the problem. This episode (organized in four segments) shows how Jane and Elayne moved back and forth between their own tentative understandings (made safe through *epistemic* and *ontological* distance, in Segments 1 & 3) and ideas presented from an external authority (at a further *rhetorical* distance, in Segments 2 & 4).

Segment 1 focused on understanding when negation would be required for the proof's **justification**. Early in the conference, Jane stated that she doubted her draft solution, because she had not used the negation of the definition of uniform continuity in her proof.

Segment 1.

1. Jane: I think my proof was fine, but I've only shown that it is uniformly continuous on [0,1]
2. Elayne: Okay.
3. Jane: I don't think it's continuous over [all real numbers], but I didn't use the negation, yeah, I didn't do this part, of showing, proving it through negation.

By listening to Jane's tone of voice when she said "fine," it was clear that she was indicating that she felt she had an acceptable proof, but not an excellent proof. Thus, this statement created *epistemic distance*, by signaling tentative confidence in her draft. Further, her statement "I didn't do this part" indicates that she recognized her attempted proof was incomplete. By describing what she did on paper, rather than stating her thinking directly, she created *ontological distance*, because it was safer to discuss why her previous **justification** was lacking, as compared to her current thinking. These two conversational moves created a productive environment for the student to discuss their emergent understandings.

A nearby student, Mindy, then interjected, asking whether or not negation was needed for the second part of the problem, noting that $f(x) = x^2$ was "continuous on all reals." Moving into Segment 2 we see the role of *rhetorical distance*, as the students appealed to the professor as an authority. In this segment, the students reached a consensus on when to use the definitions and negations of the definitions of continuity and uniform continuity.

Segment 2.

1. Jane: Kelly [another student] was asking about that. [The Professor] said if it's true [that the function is continuous on all reals] then you don't have to prove the negation…[I]n [part] 2 I showed that it was continuous over all the reals, and that's true, so there's nothing to negate.
2. Elayne: Okay.
3. Jane: But here [on part 3] I don't think it's true for all the reals, so I need to-
4. Elayne: Prove the negation.

Here, the students agreed that the negation of the definition of continuity would not be required, because the function $f(x) = x^2$ was continuous for all real numbers, so it was unnecessary to show it was not continuous somewhere. In contrast, the students recognized that $f(x)$ was not uniformly continuous for all reals, so they would need to use the negation of the definition to prove that it was not uniformly continuous on the appropriate intervals. This focused on **justification**, that is, what exactly would be needed to justify the intervals of uniform continuity and the intervals over which the function was not uniformly continuous. Here, multiple types of *rhetorical* distance helped the students coordinate their emergent thinking with an external authority.

Segment 3 focused primarily on the **logic** of the argument*,* as the students attempted to negate the definition of uniform continuity. Because the students were unsure of the appropriate **logic**, we see multiple conversational moves where Elayne creates *epistemic* distance. This happens in line 1 when Elayne says "right?" to convert a mathematical assertion into a suggestion/question, and in line 5, when she says "yeah?" after suggesting that the quantifiers should be flipped. In this way, couching her mathematical ideas as questions allowed Elayne to participate more safely.

Segment 3.

1. Elayne: 'For every' means for all, right? 'For every' means there's some [in the negation]. Right? Wouldn't it?
2. Jane: So the negation, yeah, 'for every' means there's some [Elayne annotating the negation].
3. Elayne: There's some. And then it doesn't exist-
4. Jane: Yeah, does not exist.
5. Elayne: Such that- these ones we flip them [referring to quantifiers], yeah?
6. Jane: Ooh, I don't think so.
7. Elayne: I think I remember something like that from last time.

Here, Elayne clarified the **meaning** of "for every," asking if it has the same mathematical status as "for all." The students then continued to discuss the **logic** of how to negate the definition. Elayne suggested they "flip" the quantifiers (swapping "for every" and "there exists"), which she "remembered" from a previous class period. Jane disagreed, saying "Ooh, I don't think so." As the students worked out the negation, Elayne annotated Jane's paper with the negated definition for later reference.

At this point, the Professor came over to the students' table. Jane mentioned that they "have a hunch," how to do the negation, but they want to make sure the logic is correct. The Professor made comments that reinforced Elayne's suggestion for "flipping" the quantifiers, according to the procedure learned earlier in class. As the Professor left, the students continued to work into Segment 4. Again, we see that introduction of an idea from an authority (a higher *rhetorical distance*) supported the students to return to more tentative sensemaking at lower levels of *rhetorical distance* (i.e. their own work).

Segment 4.

1. Jane: This reminds me of the keywords approach that we, like, tsk tsked.
2. Elayne: See how here they have, we have, this is like the original one. But then here it's flipped and the equals added. Yeah, so I remember that from last time.
3. Jane: Oh, good call, good call, okay.

When Jane said they "tsk tsked" the "keywords approach," she was describing how she previously rejected the method taught by the Professor. Jane's statement "good call, good call" indicates that she now realizes that Elayne's initial suggestion was a productive one, even though she initially disagreed with it. This highlights how at the *rhetorical distance* of a peer, students are more likely to be able to disagree with a claim, but that disagreement can change over time. In both Segments 3 and 4, the students discussed the use of a previous technique, which signifies a **holistic** aspect of their sensemaking.

Although neither Jane nor Elayne correctly solved the problem, there was still progress made during their peer conference, which resulted from them having space to struggle with and make sense of the proof. For instance, in Jane's initial draft, she wrote that,

> I suspect that $f(x) = x^2$ is not uniformly continuous on R based off of result from part 2, that delta depends on x_0. Informally, there will be a delta that holds that for every x_0.

Here, Jane argued that because the delta interval will depend on where on the real line the input (x_0) is, it will not be possible to find a single delta to show uniform continuity over all real numbers. Jane then sketched out a proof of uniform continuity on [0,1] showing how to bound $|f(s) - f(t)|$ using |s| and |t| and then stated,

> However, since we cannot restrict |s| and |t| over the reals, so *f(x)* is not uniformly continuous over R.

Essentially, her informal argument was that because the argument used to prove uniform continuity on [0,1] could not be used over all of R, *f(x)* could not be uniformly continuous on R. Of course, this reasoning is flawed (even though $f(x) = x^2$ is, in fact, not uniformly continuous on the reals). In her revised work (post peer conference), rather than the informal argument, Jane correctly negated the definition of uniform continuity and attempted to prove the intervals over which $f(x) = x^2$ was not uniformly continuous. Nonetheless, she got stuck and never constructed a correct proof.

Jane provided written feedback through annotations of Elayne's initial draft, which also provided some insight into her sensemaking. In multiple cases, she noted where Elayne needed to "specify the domain," or clarify "x, x_0 *in D*." She also suggested that for the proof of uniform continuity, that Elayne should "choose a <u>closed interval</u> that is <u>bounded</u>." This particular comment provides deeper insight into Jane's thinking, because in her own written work she never attempted to prove where *f(x)* was uniformly continuous, only that it was not uniformly continuous on all reals. As we can see based on Jane's written feedback, even though Elayne did provide productive assistance to Jane during the conference, Jane was still able to see meaningful areas of improvement for Elayne's work. This is the value of a partially correct attempted proof used in peer review.

In summary, the *ontological* and *epistemic* distance created in Segment 1 created space for Elayne and Jane to engage with **meaning**, **logic**, and **justification** without fear of being wrong. Later, in Segment 3, Jane disagreed with the suggestion of her peer, until Segment 4, when the same approach was advocated by the Professor. In this way, Jane deflected the suggestion of her peer who had less authority, and then later accepted the external authority (related to *rhetorical distance*). This highlights how *rhetorical distance* mediates a students' willingness to challenge a claim. Thus, we see how *epistemic* and *ontological* distance were used to support tentative sensemaking in Segments 1 and 3, and then *rhetorical* distance was used to support validation from an external authority in Segments 2 and 4. The peer review context made this back-and-forth possible in a seamless way.

Episode 2: Using written work to externalize ideas

The second episode (broken into five segments) provides multiple examples of how *ontological* distance created a productive setting for sensemaking, as Landon and John referred to their externalized ideas. In Segment 1, John describes how he *wrote* his work (past tense emphasized), and in Segments 2 and 4 the students discussed Landon's annotations of that work. This set them up for a later conversation with the Professor in Segment 5, where they could refer to their externalized work.

Segment 1.

1. Landon: I see you have uniform continuity applies to a single epsilon band across the entire function. So, when you say a single epsilon band across the entire function simultaneously, it's- you're saying that we're still focused on the point-
2. John: But those- you're focused on two points, but those points are both allowed to float, is the way I wrote it.
3. Landon: Oh, okay.
4. John: Those points- it has to be able to hold- so basically you can say I want you to be- the challenger can say I want you to be within this epsilon, I want the distance between any two y values to be within this epsilon distance, or be less than this epsilon, and then you can respond for a difference between x values that will get you there, but for any two y values. So, you're supposed to be able to limit the dist- at least that's how I read it, and I might be way off on that but that's how I'm reading it, is that it's saying that-

Here, John described a particular way to make **meaning** of the definition. He conceptualized a challenger giving him an error tolerance epsilon, to which he needed to respond with an appropriate delta. When he referred to "a single band," he meant that the error tolerance was something that was applied to the function globally, rather than locally at each point as with continuity. John created *ontological distance* by describing the way he "wrote" his work, allowing for John and Landon to discuss the meaning of uniform continuity. As John provided an extended explanation, he added the hedging language "I might be way off but that's how I'm reading it," which created *epistemic distance.* In Segment 2, both students tried to make deeper **meaning** of this conceptualization by thinking about what would happen in the case of an exponential and a quadratic function (the latter of which was written on John's initial draft as an annotation by his partner Landon).

Segment 2.

1. John: Your $f(x)$s are going to be really far apart while your xs are going to be close to each other.
2. …
3. Landon: Just from like here to here our fs are pretty close.
4. John: Right, right.
5. Landon: Right? But then as we go out, say this one, our xs are going to be quite a bit farther away.
6. John: Yeah. Which is actually something similar, less drastic, but that happens with x squared.
7. Landon: Right, this is- Yeah, so what I've done is say this is the x squared graph as an example, that the distances between $f(x)$s are-
8. John: Right.

9. Landon: -going to be changing. So are we trying to say that maybe the *f*(*x*) equals *x* squared is not uniformly continuous?
10. John: Yeah, I would argue that. I mean the function itself is definitely continuous.

Here, the students pointed to various regions of the exponential function and quadratic function to coordinate how the changes in the function's output value differed depending on the location of the inputs on the *x*-axis. Landon's annotation provided another form of *ontological distance*, because it allowed Landon to externalize his idea onto the draft, which could then later be constructively critiqued by both students. This critique resulted in the development of a visual argument, which led the students to assert that $f(x) = x^2$ would not be uniformly continuous on *R*, because the graph becomes increasingly steeper as *x* increases. Still, Landon's question "are we trying to say…" created a form of *epistemic distance*. It signaled that the students were coming to an emergent consensus, but it would still be acceptable to change their minds later.

In Segment 3, the conversation shifted from using this intuitive understanding to actually constructing a formal proof.

Segment 3.

1. Landon: Coming up with a delta is kind of an odd situation.
2. John: Right.
3. Landon: I actually did that on my last [peer review problem] because [$f(x) = x^2$] was one of the functions that I chose. So my delta was also in terms of epsilon.

Here Landon referred to work that he did to find a delta interval for proving the continuity of $f(x) = x^2$ in a previous problem, which involved finding a delta interval that depended on epsilon. Thinking **holistically**, he was now aiming to do something similar on the current problem. As the conversation continued, the students spent a considerable amount of time working through the mechanics of how to find a delta interval. Much of this scratch work was written on John's initial draft by Landon (again, related to *ontological* distance). The students then revisited their intuitive **meaning** making in Segment 4, by looking at input values *x* and *c* for the function.

Segment 4.

1. John: But in this context is that really what's happening?
2. Landon: Well I don't think that *c* is approaching *x*, what we're trying to say is that no matter how close *c* gets to *x* we'll always be able to have a distance- we're always going to have another, like, [region where] the *y* values that are within an epsilon distance.
3. John: Okay.
4. Landon: Right, because generally if someone were to ask you is this continuous at this point, you'd be like yeah. But as we, you know, as we move, say, I don't know, to *x*, if this is *x* plus *c*, as we go this way it's like we're going to keep finding values and it's fine. But if this was *x* minus *c* eventually someone's saying that if you're trying to argue that it's not continuous, you're going to eventually find a delta-

Here, Landon attempted to clarify the **meaning** of *x* and *c* in his formal argument, describing how for a continuous function it would be possible to bound the change in the output, but for a function that is not uniformly continuous, there would eventually be a delta for which the change cannot be bounded. (In fact, it would be impossible to bound the change in output no matter which delta was chosen, because the delta interval could simply be positioned around sufficiently large values on the *x*-axis). After this discussion, the Professor came to the students, and in Segment 5, the students tried to make deeper **meaning** of uniform continuity with him.

Segment 5.

1. John: We both feel like we have a pretty strong understanding of continuity…But uniform continuity we're struggling with a little bit…I'm struggling with a way to show where- because I feel like with *x* squared, for example, that *y*'s getting larger and larger, further and further apart.
2. Professor: Right.
3. John: So because it eventually- I mean I'm trying to find a way, can we just choose a really big one and it'll still work for the small one?
4. …
5. Professor: What's going on with the continuous case is if someone gives you a point you can bound the amount of change by staying within [a particular region]. So, it's sort of like locally the steepness is bounded there. But the problem is, like, x squared, the steepness is not bounded. You can get [$f(x)$] as steep as you want by simply going up far enough [in *x*].
6. …
7. Landon: Right, the distance between the $f(x)$ and $f(c)$ is going to grow depending on how far *x* goes.
8. John: Right

In this segment, the students expressed their confusion in line 1 when they described they were struggling (creating *epistemic distance* from their tentative claims), and also suggested a possible path forward to the Professor. The Professor validated the students' idea (from the *rhetorical distance* of an authority), and this idea was ultimately what the students wrote in their submissions. In John's draft, he only had brief scratch work that was intended to support a proof that $f(x)$ was uniformly continuous on [-1,1], but he didn't offer a formal proof. In the revised work, John provided a mostly correct proof of uniform continuity on [-1,1]. He also attempted to prove that $f(x)$ was not uniformly continuous on $(1, \infty)$, and $(-\infty, -1)$. Although he was able to use the correct negation of the definition, he got lost in the mechanics of the proof and did not complete it. In addition, John did not realize that x^2 would be uniformly continuous on any closed and bounded finite interval.

Again, we summarize how distance was relevant to sensemaking. In Segment 1, John described what he wrote in his work (*ontological distance*) and created *epistemic distance* from his assertion, stating "that's how I read it, and I might be way off." Throughout the interaction, John and Landon built on each other's ideas as relatively equal peers (facilitated by their *rhetorical distance* from one another's ideas). This was supported by Landon's annotation, which created *ontological distance* from the idea, allowing both students to constructively critique it. As the students worked through the ideas in Segment 2, John took up Landon's idea that $f(x)$ would not be uniformly continuous due to the increasing steepness of the function. Finally, in Segment 5, the role of the Professor as an authority (at greater *rhetorical distance*) helped the students feel more confident to follow through with their proposed approach.

Quantitative Results

The above two episodes highlighted the prevalence of multiple forms of sensemaking across different conferences. (The first episode highlighted **justification**, **logic**, and **holistic**, while the second episode highlighted **meaning** and **holistic**.) Next, we performed quantitative analysis to capture the prevalence of sensemaking forms (see Table 3).

Table 3. Student sensemaking of proofs (N=20 conferences).

Sensemaking practice	Number of conferences containing this practice
Meaning	17
Logic	10
Justification	20
Holistic	11

We hypothesize that these practices were prevalent across a variety of conferences given the nature of peer-review conferences. Because students entered the conference with their own attempted proofs, it created an *ontological distance* that allowed the students to easily use their existing work as a starting point for developing deeper understanding. This contrasted the more free-form nature of group work, in which students are collaboratively constructing the solution together, and the draft work is not distinctively positioned as an external artifact. In addition, the inclusion of ideas from sources at a variety of *rhetorical distances* allowed students to coordinate both emergent ideas and ideas that are assumed to be correct, while *epistemic distance* created space for more tentative discussion, pushback, and revision.

Results: Perceptions of Proof

The results in this section address the following research question: *How are forms of distance and sensemaking practices related to students' perceptions of proof?* The following analyses focused on how students developed perceptions of proof, both through the present course and their prior courses. The results indicated that students developed different perceptions based on their engagement with peer review, as compared to their engagement in prior courses. A summary of codes is given in Table 4.[2]

Table 4. The number of essays containing each type of student perception (N = 13 essays total).

	Peer Review	Prior Courses
Connection	11	1
Exploration	8	1
Alienation	-	9
Memorization	-	9

Peer Review Experiences

Students described how their sensemaking in peer-review conferences influenced their perceptions of proof. By coordinating various forms of distance in the sensemaking process, students were able to safely explore the tentativeness of back-door mathematics. Students

[2] A single student discussed deep engagement in a prior inquiry-based proof course. This was the only instance of connection and exploration in the Prior Courses category.

described their proofs as something that belonged to them (a lower level of *rhetorical distance*), rather than an external authority.

Connection. Peer review helped students feel connected to their own proofs. By expressing mathematical ideas both formally and informally (e.g., through their conferences, or by using a two-column format), students could create *epistemic distance* that allowed them to safely explore mathematical proofs. Similarly, the creation of drafts created a productive *ontological distance*, as students were sanctioned to engage with the same proof multiple times.

John described how peer review had "completely changed the way [he] approach[ed] proofs," making proof "feel personal." He attributed this to the "annotations column and its free form" that gave him "a voice and choice in proof that didn't exist prior." Similarly, Elayne discussed how she could "understand and interpret" her own proofs,

> I think it was very helpful to not have an emphasis on simply providing a correct proof, rather on providing a proof that I was able to understand and interpret. It made me see writing proofs as something that I am capable of doing.

The opportunity to receive feedback and revise removed the pressure of "simply providing a correct proof," which allowed Elayne to instead focus on sensemaking. Matt also discussed how peer review gave an opportunity to "at least try to make sense of what [he] was trying to prove." Matt acknowledged that his attempted proofs might not be completely correct, but at least he "could create a logical argument" and use proof to "create new meanings and understandings."

Exploration. The iterative peer review process highlighted the tentative nature of proof. For instance, Mindy described how it helped her "figure [a proof] out" on her own, rather than "look it up online." Similarly, Melvin contrasted the goal of "proving things" with simply having a "proof itself." He considered peer review as "a big part of the reason [he] looked at proofs differently in this course." John explicitly discussed the role of revision, and how it removed the fear of failure,

> Having the opportunity to review a draft with a peer and then revise my work has taken the fear of failure out of the process. How does one fail a draft? The process also makes my own growth undeniable when I compare many of my drafts to their revised counterparts.

Mindy, Melvin, and John all highlighted the importance of making sense of a proof, rather than just having a proof. Because they could attempt proofs, receive feedback, and revise, this sensemaking was sanctioned as an important part of the course. John's statement, in particular, draws attention to the importance of *ontological distance*, which can be created by the unique nature of having a draft solution as an artifact to engage with during the peer review conferences.

Prior Experiences

Students associated their prior experiences in proof-based courses with alienation and memorization. In this context, proof was perceived as belonging to the teacher or textbook (a larger *rhetorical distance*), which contributed to feelings of a lack of ownership.

Alienation. Landon expressed frustration with disconnection from “elite” mathematicians,

> Although book-publishing mathematicians enjoy entertaining the elite few who do not toil underneath the deceptively succinct, clever, and often counterintuitive prose that lies between Proof: and QED, undergraduate and graduate level mathematicians need a more accessible way to eat their proof pudding, so to speak.

Landon’s description of “entertaining the elite few” painted a picture that mathematics is done by people who are not like him. John also described frustration with the expectation he should “conform to rigid conventions and opinions of elegance.” John was careful to emphasize that proofs should not “become less rigorous,” but that there was a need to “delete the exclusionary elitism” from proof instruction.

Kelly described a perceived arbitrariness to grading in her prior courses, and the need to conform to what “the professor expected.” Nicola also described her experience with a student grader who would give “extremely low homework grades” because “he didn’t like” the way she wrote her proofs. Her perception was that she wrote “too much and he wouldn’t even try to make sense of it.” These experiences suggest that while students may have been exposed to front-door mathematics, they may have had insufficient opportunities to tentatively explore back-door mathematics in the making. The discussion of mathematics as belonging to an external authority highlights a high level of *rhetorical distance* between students and their proofs.

Memorization. Students described memorization as a key strategy to succeed in prior courses. For instance, Mindy memorized proofs “without really understanding” what she was memorizing, and if she “had to write [her] own proof,” she would “first find it online.” Similarly, Brigette described how she would “memorize the proofs in order take tests without understanding questions, steps, and answers.” In both of these cases, the proofs did not really belong to the students, but they were ideas copied from an authority (high *rhetorical distance)*. Other students, like Olivia, described how memorization was reinforced by their instructors,

> The professor said, “All you have to do is memorize all these proofs and you will be fine.” In fact, I passed Calculus solely on the fact that I have the skill of regurgitating information.

Mindy’s reflection draws attention to the explicit ways that learning environments might promote a memorization approach, which complements the subtler implicit messages that a learning environment might convey.

Limitations

While our study focused on opportunities for sensemaking and resulting changes in perceptions of proof, our study design did not allow us to document the extent to which students improved their understanding of real analysis concepts. Moreover, given the variability in content areas (e.g., space-filling curves vs. Fourier series) and homework problem difficulties, it was not possible to make claims about student growth in sensemaking practices. In addition, because the first author was the instructor in the course, there may be some bias in how the students described their experiences with peer review. However, given that student responses to the anonymous course survey (submitted to the university) were also uniformly positive, the impact of this bias is likely small. We also recognize that student perceptions of mathematics were due both to peer review and the collaborative nature of the course, and it is difficult to disentangle the two. Lastly, we recognize that there is some level of subjectivity in interpreting students' statements, and we cannot be 100% certain of our interpretations of student talk.

Summary and Implications

This study connected sensemaking practices and perceptions of proof through student engagement in peer-review conferences mediated by various forms of distance. Peer review conferences provided a context for students to engage with idea from an authority, from peers, and their own ideas (i.e. multiple *rhetorical distances*). *Ontological distance* between students and their drafts provided a productive basis for conversations, and students created *epistemic distance* to safely work through their tentative ideas as conferences unfolded. This study makes important theoretical contributions to the study of proof, both by introducing the new concept of *ontological distance*, and also by empirically documenting the role of various forms of distance in sensemaking of proofs. Our model adds complexity to the conceptualization of proof comprehension and validation, by highlighting the role of the social environment in which students learn to make sense of proofs. This research has at least three implications for mathematics education.

First, we found that the majority of peer review conferences contained all four sensemaking practices (**meaning, logic, justification,** and **holistic**). This suggests that peer review conferences can be a productive venue for students to learn to make sense of proofs. This has pedagogical utility for instructors teaching proof and also for researchers who would like to study the development of proof sensemaking practices. Although it was beyond the scope of this study, we believe that studying the growth of student sensemaking practices over time is a productive area for future research.

Second, we provide insight into the role of distance in sensemaking. When students intentionally created distance from particular claims, they created a safe space to explore their partial understandings (i.e. back-door mathematics). Then, by strategically incorporating ideas from an external authority (i.e. front-door mathematics), students were able to validate their ideas while still retaining ownership over their thinking. By coordinating front-door and back-door engagement, students were able to develop a more positive perception of proof. By design, peer review conferences easily incorporate a variety of forms of distance in a way that differs from standard group problem solving or traditional lecture alone. Of particular importance is the existence of a draft solution, which supports the creation of *ontological distance.* This new construct can also be generalized to other contexts. For instance, it could be used to understand the use of other artifacts, such as whiteboards, and how they might impact student sensemaking.

Third, the student essays highlight that these students completed their undergraduate degrees in mathematics and pursue graduate studies *in spite* of their negative experiences with proof. While prior research highlights how mathematics can alienate students and push them towards studying other subjects (Ernest, 1992), this finding suggests that even students who do choose to continue pursuing mathematics may feel alienated. It would be worthwhile to understand in future studies why students like these chose to persist. Especially given that most participants in the study were either practicing teachers or had future aspirations to teach mathematics (at the secondary or post-secondary levels), it is paramount that mathematics education also serves to inspire them, so that they can also inspire future generations of learners.

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