Five practices for supporting inquiry in analysis

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Abstract: This article describes an inquiry-oriented real analysis classroom in which students were guided to discover mathematics for themselves. To support student inquiry, the framework of "five practices" from K-12 education was used. To illustrate this framework, two case examples are given from actual discussions that took place in this classroom. These examples focus on student construction of sequences of functions, both the space-filling Z-order curve and the Cantor staircase function. The article closes with practical guidance for other instructors who would use the five practices in their higher-level mathematics courses.

Biography:

Dr. Reinholz specializes in STEM education transformation with the aim of increasing equity and diversity in STEM fields. His work is grounded in a holistic design perspective, which draws on research in disciplinary learning, equity, and organizational change. At the classroom level, he focuses on how reflection and peer feedback can deepen disciplinary learning. Beyond the classroom, he studies how cultural and structural features of higher education can support and inhibit meaningful transformation. Through his leadership in the Access Network, he works to improve access to meaningful STEM learning across the US.

Introduction

This paper describes the application of the "five practices" framework for orchestrating discussions [1, 2] to higher-level mathematics. In the same way that a skilled politician can anticipate the outcome before they call an issue to a vote, an instructor needs to know what contributions students are likely to make before bringing the class together for discussion. Such knowledge allows an instructor to weave together student contributions in a way that they meaningfully build on each other. The five practices are concrete tools that an instructor can use to anticipate and guide a discussion along a desired course.

Developed in K-12 mathematics classrooms, these practices are still extremely relevant to higher-level mathematics. This article illustrates the five practices using two actual lessons taught by the author that took place in a graduate-level analysis course. It closes with practical guidance for other instructors to use these practices in their own classrooms.

Introducing the Five Practices

Classroom discussions are a critical part of learning [3–7]. But simply speaking is not enough; students need to engage in high-level, cognitively-demanding discourse [8, 9]. Given the complexity of managing social interactions, subtle inequities often emerge in who gets to participate and how they get to participate [e.g., 10–12]. Described below, the five practices are one set of techniques an instructor can use to support high-level discourse and help address these inequities, by carefully choosing which students get to share their ideas and how the ideas are taken up by the class.

The five practices are: anticipating, monitoring, selecting, sequencing, and connecting [1, 2]. The practices are generally used in a lesson structure that consists of three phases: launch, exploration, and discussion [2]. During the launch phase, the instructor provides relevant background information and orients students to the mathematical task at hand. The exploration phase consists of students working together (typically in small groups) for the bulk of the lesson. The lesson ends with a discussion that allows students to share their thinking across groups, with the support of the instructor, to tie the mathematical ideas together. While the five practices are tailored for such a lesson structure, they are general tools that can be used in a variety of settings.

The first practice is *anticipating*. Before teaching a lesson, an instructor *anticipates* possible student responses to a task. The first time a lesson is taught, an instructor may have limited insight into how students will engage with the relevant mathematics. Talking to other instructors or consulting the research literature can be helpful, as time permits. The second time a lesson is taught, anticipating becomes easier. When an instructor teaches a lesson, they can save samples of student work and keep fieldnotes describing areas of struggle and insight, which support anticipation for the next iteration of the lesson. The lessons described in this paper were both taught for the first time by the author, so anticipating student thinking was a challenge. In brief reflections after the lessons, I describe how the lessons could be adapted when used again.

The second practice is *monitoring*. This practice takes place while students are exploring the mathematical task. In addition to helping students progress on the task at hand, an instructor also monitors student thinking, taking notes that describe: their approaches, what false starts they have, their productive breakthroughs, etc. Logistically, this can be achieved by carrying a notepad or clipboard during instructional time to write down what students are doing. If an instructor uses this approach, they should communicate to the students why they are taking notes, to make sure that students feel comfortable. The purpose is not to judge student work, but to help document their productive thinking so it can be shared with the class.

The third practice is *selecting*. As an instructor becomes aware of the ways in which different groups of students are thinking, the instructor must choose which student ideas to highlight in the plenary discussion. This could be any combination of struggles, insights, or alternative solution methods. When selecting particular student ideas, the instructor should give students an advance notice that their ideas will be showcased in the plenary discussion. By encouraging students to share, and asking for permission to use their ideas, an instructor can make it more comfortable for a greater variety of students to speak in the plenary. Giving an advance notice in this way avoids putting students on the spot. This is one small way that an instructor can use the five practices to help encourage greater equity in classroom participation.

The fourth practice is *sequencing*. Beyond figuring out which ideas to share, an instructor chooses the order in which they will be shared. One strategy for sequencing student responses is to begin with more concrete solution strategies and move up levels of abstraction. Another strategy is to begin with specific examples and move to general arguments. By appropriately sequencing student responses, an instructor opens space for more students to share their ideas with the class. This provides opportunities to elevate the status of lower-status students [13], because they can be given opportunities to meaningfully contribute to the discussion even if they do not have an entirely complete solution. In contrast, if the first student to share their thinking gives a complete analytic solution, it tends to shut down further conversation and leaves little room for students to discuss false starts, potential paths that did not follow to completion, and so on. Of course, carefully sequencing student ideas is insufficient to address issues of classroom equity, but it is one useful tool.

The above four practices all take place before a discussion begins. If an instructor has been successful in monitoring, selecting, and sequencing students to share their ideas, a productive discussion is much more likely to result. The fifth practice is *connecting*. As students share their ideas, the instructor helps explicitly bring the ideas together, highlighting how the ideas may build on one another or contrast each other.

Method

This paper focuses on an inquiry-oriented analysis course taught for the first time by the author. This graduate-level course was designed for students in a teaching-focused Master's degree program in the US. A total of 13 students were enrolled in the course, all who consented to participate in the study. There were five in-service teachers, five Graduate Teaching Assistants (GTAs), and three prospective teachers who were not yet teaching. All student work was collected, but classroom sessions were not video recorded. As such, the primary data sources for this article are student work, lesson plans, and instructor notes written during and after class.

Students worked in teams organized by principles of team-based learning, including static groups, team-based assignments, and team contracts [14]. There were four teams in the class: The Mathletes, The Angles, Function Families, and Cauchy's Island. These names were chosen by the students, and are used to refer to the work of different groups throughout the paper below. As teams, students used a variety of media to engage in mathematics, including whiteboards, sidewalk chalk, and construction paper, as appropriate (see Figure 1 for an example of students working together on an inquiry task). Students also engaged in Peer-Assisted Reflection [PAR; 15], a structured review process, but it is not the focus of this article.

<Insert Figure 1>

In the following two sections, I introduce the relevant mathematics to situate the discussion of student thinking. I then describe how I launched the task and discuss my use of the five practices to facilitate classroom engagement. Finally, I close with brief reflections on lessons learned and possible modifications to the lessons.

Space-Filling Curves

Launching the Task

The first activity focused on student constructions of the Lebesgue curve. Like other space-filling curves, the Lebesgue curve can be defined as the limit of a sequence of functions that cover a unit square by repeatedly subdividing the square (by fourths in this case). To construct a map that converges, the curve must traverse the unit square through these subdivisions along an appropriate path. The path for the Lebesgue curve represents a Z-shape, so the Lebesgue curve is also called a Z-order curve.

The Lebesgue curve also provides a mapping from the Cantor middle thirds set to the unit square. This is done by associating a binary expansion with each subdivision of the square. Because the Cantor set consists of all ternary expansions containing only the digits 0 and 2 (the 1's are the middle thirds that are successively removed), the Cantor set can be described in its entirety using binary expansions, where all 2s are replaced by 1s. Thus, students began the problem with two representations at their disposal: a Z-shaped path for the curve, and binary expansions associated with each subdivision of the square. See Figure 2 for a student illustration of this pattern.

To launch the exploration, I introduced the Lebesgue curve by demonstrating how the mapping works for four subdivisions of the unit square. Students were provided with the binary and decimal numbering of the squares, as in Figure 2. I also provided the next iteration of the sequence, with the curve drawn and binary expansions written (but not the decimal equivalents).

Students were told to find the next members in the sequence of functions, and once they had discovered how to construct an arbitrary member of the sequence, they were told to explore the properties of this sequence of functions. As was typical in this class, students were guided by their instructor to explore different mathematical properties through an open-ended task. The five practices are highlighted below in *italics*.

<Insert Figure 2>

Using the Five Practices

Although I had never taught this task before, in previous lessons I monitored students working with the Cantor set, ternary expansions, and the Hilbert space-filling curve, so I had some idea of how they would engage in the present task. During those prior lessons, I had learned that only a few students (e.g., Melvin in the group Mathletes) had much experience working with binary. For this reason, I assumed that students may need extra support with binary. One concrete way this manifested in my teaching was that I provided students with both the binary and decimal descriptions of each subdivision of the square (as described above), even though the decimal expansions were mathematically unnecessary.

During student work time, I circulated around the room to *monitor* student thinking. To begin, most of the student groups tried to convert the binary expansions in the second-order curve to decimal. Function Families noticed that the binary expansions were just counting up as one followed the curve through the 16 subdivisions of the unit square, so rather than actually performing the conversions, they just filled out the numbers 1 to 15 in the subdivided squares. Other groups soon noticed this too. In contrast, the Angles looked at binary expansions moving from left to right (like in reading), rather than following the curve. Thus, they did not notice this pattern and instead individually converted each of the expansions from binary to decimal. If the group was to continue in this way, it would be a long, tedious task, and they would not actually have a chance to grapple with the meaningful mathematics I had hoped them to spend time with. Accordingly, I mentioned to the Angles that I had seen other groups looking at the expansions in different orders, and suggested that the Angles might also see if they could find any patterns to simplify their work. I *selected* this thinking as something to return to later, to highlight a general problem-solving strategy that students might use (following the curve to think systematically).

As I monitored the groups, I worked out an order to sequence student ideas. I noticed that some groups (particularly Cauchy's Island) had quickly moved to more general arguments while other groups were still making sense of the problem situation. Thus, I decided to sequence student ideas according to the generality of the approach, so that students could see the work of one group as building on the next, rather than shutting down discussion by looking at the "best" answer right away. I present student ideas below in the order that I chose to sequence them, not the order that I encountered them in.

The Mathletes spent a lot of time trying to draw out the third-order function (with 64 squares), without even looking at the binary expansions in the squares. Without binary expansions to help identify different subdivisions, the group made a number of errors due to the complexity of the drawing and was unable to find a path to connect the squares. After I discussed their approach with them, the Mathletes eventually developed a strategy to determine the numbers for each of the 64 squares in the third-order curve. This supported them to successfully draw the function. Ironically, Melvin, who was familiar with binary was a member of this group, yet his group chose not to use binary, which resulted in difficulty.

Function Families focused on the 1-2-3-4 pattern that the first-order curve used to traverse the four subdivisions (top-left, top-right, bottom-left, bottom-right). They then noticed that this same pattern was recursive within the next iteration (the second-order curve). In other words, they noticed that to move from one iteration to the next, they had to subdivide each of the existing squares into fourths, and draw a "Z" within the new boxes that provided four new points. Using this idea, the students drew the third-order curve without even looking at binary expansions: they just continued working in decimal. This was an idea I wanted to highlight to the students, to show how they could look for patterns to simplify their problem-solving process.

The final group, Cauchy's Island, developed a method to conceptualize the binary digits using compass directions. They started with the case of two binary digits (four squares). If the first digit was a 0, it meant that they were on top half of the grid (north), and if it was a 1, it was the bottom half (south). The next digit indicated the left side (west) with a 0, and the right side (east) with a 1. The students generalized this to the second-order curve, by thinking of nested subdivisions. The first pair of binary digits was used to describe the largest subdivisions, the second pair described another layer of subdivisions, and so on (see Figure 3).

<Insert Figure 3>

To test their method, Cauchy's Island attempted to locate the position of an arbitrary six-digit string of binary digits using this idea (see Figure 4). The first step was to draw the red line in the center of the grid. Using the two-digit compass directions, one of the four quadrants separated by red was selected (i.e. first choosing north/south and then east/west). The quadrants were split two more times, first in green, then in pencil, and the location of the six-digit binary string was located. Satisfied that this test worked, Cauchy's Island worked through this process forward and backward: given a binary expansion, they could locate its position, and given a particular position, they could determine the corresponding expansion.

<Insert Figure 4>

In the synthesizing discussion (~15min) that followed student explorations (~45min), I *sequenced* student ideas as follows: (1) the Mathletes drawing the function without using binary expansions, (2) how the Angles performed their conversions in order, (3) the 1-2-3-4 patterns that Function Families noticed, and (4) the compass directions of Cauchy's Island. To guide the discussion, I told groups in advance what they would be asked to share. I sequenced ideas in this order to start with initial explorations and finally move to a complete solution at the end, while still having each group share some of their productive ideas.

I *connected* these ideas in the final discussion by highlighting different strategies that students used, looking at both areas of struggle and success. Having students share their strategies also moved away from simply looking at right and wrong, or who got the "best" answer. Instead, groups were able to share the issues they got stuck with (e.g., The Angles not choosing a systematic order for conversions) to highlight strategies that other groups might consider in the future. In addition, I shared another method for mapping binary strings to the square, which was conceptually distinct from the idea shared by Cauchy's Island. The purpose of sharing this method was not to tell students that it was the "correct" answer, but to demonstrate that there were many ways of describing this mathematical situation. Because the students in this course were teaching lower-division math courses (either in K-12 schools or college), I connected the various approaches and false starts to developing more productive problem-solving strategies.

Reflection

Because this was the first time I had taught this task, it was difficult to anticipate how students would engage. Looking at student engagement showed a variety of false starts and dead-ends in problem solving. It might be tempting to provide more guidance in a future iteration of the lesson to curtail this exploration time. However, I feel that it is important for students to explore in this way to develop as problem solvers, so I would not change that aspect of the lesson.

Still, there are possible modifications for the future. For me, the greatest surprise was the strategy used by Cauchy's Island. This mathematically-valid strategy was distinct from what I had encountered in solving the problem myself and in reading other resources online. In teaching the lesson again, I could envision an extended version of the exploration that would have students find multiple different mappings and compare and contrast them. This would extend the lesson from one to two days.

Another connection that did not come out in this lesson was between the Cantor set and the Lebesgue curve. Even though prior lessons explored the ternary representation of the Cantor set, no students drew upon this prior knowledge, and I did not make it a focus of the discussion. Thus, I could envision another alternative version of this lesson in which I provided more discussion of ternary and the Cantor set (or even added new opening activities), and then had students think about mapping the Cantor set to the unit square.

Cantor and Lebesgue Functions

Launching the Task

The second activity focused on Lebesgue function (also known as the Cantor function). For the purposes of the activity, students were introduced to two “different” functions that were called the Cantor and Lebesgue functions. The idea was that through exploration the students would come to see that these two functions were actually the same. The Lebesgue function was described as an iterative sequence (see Figure 5). The Cantor function was described as a mapping from [0,1] to [0,1] by the following set of directions:

1. Express x in ternary (base 3).
2. If x contains a 1, replace every digit after the first 1 by 0.
3. Replace all 2s with 1s.
4. Interpret the result as a binary number. The result is $C(x)$.

Students were provided with a link to an online base converter so that they would not need to perform conversions by hand. For this task, Mathletes and the Angles were assigned the Cantor function (the ternary expansion), while Function Families and Cauchy’s Island had the Lebesgue function (the sequence of functions). Students were not told that the sequence of Lebesgue functions would converge to the Cantor function, but the goal was that they would discover this during the plenary discussion.

<Insert Figure 5>

In addition to the definition of a function, students were provided with a set of questions: (1) What do they look like? (2) Are they bounded? (3) Are they continuous? (4) Are they differentiable? (5) Are they integrable? (6) What else can you discover?

Using the Five Practices

This exploration took place after the previous one, so students already had experience with ternary expansions and the Cantor set. It was *anticipated* that students would quickly connect the Cantor set and Cantor function, but in *monitoring* the groups, it became clear that students did not make this connection. To recover from this poor anticipation, I had to explicitly prompt the students to think about the Cantor middle thirds set. In fact, it quickly became evident that this was a very difficult task for the students. I adjusted my plan and the class ended up spending two class periods on it rather than one as originally intended. The students spent the first class period (approximately 50 minutes) building the functions and building intuition about them and spent the next period answering the questions (1)-(6) above. Of the two functions, the Cantor function was much more difficult for students to build intuition about, because the iterative nature of the sequence of Lebesgue functions offered a lower barrier of task entry.

Early work with Cauchy's Island showed that they needed support to understand the nested set of piecewise functions. Their approach was to write the third member of the sequence as a nine-part piecewise function, the fourth member as a 27-part piecewise function, and so on. This approach was quickly unmanageable, so I pressed them to think graphically instead. Similarly, the Mathletes were not sure how to approach the Cantor function. They used the converter to look at one binary string at a time, rather than thinking more holistically. Accordingly, I pressed them to think about the Cantor set and to think about how the function may differ when the input value was part of the Cantor set or not.

With guidance to look at numbers inside and outside of the Cantor set, the Mathletes were able to discover that each removed middle third would all map to the same y-value. They next discovered that each middle third would map to a y-value halfway the distance between the previously removed intervals (see Figure 6). Following this reasoning, they quickly concluded that the function must be continuous off of the Cantor set. With a bit of prompting about the definition of continuity, they discovered that the function must (surprisingly) be continuous everywhere, because the nature of the mapping *off* of the Cantor set afforded an argument to "bound the change" of the function *on* the Cantor set.

<Insert Figure 6>

In contrast, Function Families was unable to systematically use base conversions to understand the Cantor function. I considered allowing the group to continue exploring the base conversions, but I was afraid that they would never reach the point of exploring function properties, which was my true goal for the lesson. Thus, I used this as an opportunity for students to share work across groups. I had "Team Cantor" and "Team Lebesgue" form as super groups to compare their work. As the groups met together, Function Families was able to generalize beyond their representation at specific points of the function so that they could begin to explore function properties when they returned to their own group. This instructional move was used so that Function Families could move beyond the base conversions and look at some other interesting function properties, such as continuity.

The Angles also had some difficulty discovering function properties. However, they did notice that the functions could not be differentiated at the endpoints of the "thirds" of the intervals, because there were corners there. This was *sequenced* early on in student presentations so that the group would have something positive to contribute.

<Insert Figure 7>

Cauchy's Island made some interesting observations about the differentiability of the function on the intervals. They first noted that the derivative would also be zero where the middle thirds were removed and that the other portions of the curve would become increasingly steeper (see Figure 7). Because the height of the function was reduced by 1/2 as it was compressed down for the next iteration, but the width was reduced by 1/3, the slope increased by 3/2 at each iteration, eventually diverging to infinity.

My goal for sequencing student work was to have students first present about the Lebesgue function, and then the Cantor function, so that hopefully students would see the sequence of functions converging to the final result. As such, student work was sequenced as followed: (1) the Angles and differentiation on the endpoints, (2) Cauchy's Island observations about derivatives everywhere, (3) Function Families working with the inverse function, and (4) Mathletes' observations about continuity. These results were *connected* so that students could hopefully see the relationship between the Cantor and Lebesgue functions. Indeed, as Cauchy's Island was presenting their results on the Lebesgue function, Melvin exclaimed that they were converging to the Cantor function, and the whole class discussed this idea. This opened up space for the students to discuss what properties might be conserved through convergence of a sequence of functions, which led to future lessons on uniform convergence.

Reflection

The first observation I made early in the lesson was that this task was more appropriate for two class periods rather than one. In a second iteration of the lesson I would begin with one class period focused on constructing the functions and a second class period focused on function properties. I also did not *anticipate* that students did not have a systematic approach to exploring the Cantor function. In re-teaching the lesson, I would draw explicit connections to the Cantor set and help students be more systematic in their choice of points to map. On a positive note, I was pleased to see that students spontaneously connected the two different functions during the plenary discussion, without any additional prompting from me.

Discussion

The five practices are tools an instructor can use to help set the course of a discussion before the discussion even begins. They support an instructor to highlight thinking from different students, connecting and building on what various students contribute. Moreover, by sequencing student responses appropriately, an instructor can help address status issues for students who may need more scaffolding to engage with the mathematics, so that they still have an opportunity to contribute their thinking to the discussion in a meaningful way [13].

Developing inquiry-oriented lessons is a challenge. This process requires access to rich mathematical tasks, which are not always readily available, especially in higher-level mathematics courses. Moreover, the first time any lesson it is taught, it is difficult to anticipate what students may or may not do. Indeed, my initial anticipations in both of the above lessons needed to be modified. However, as an instructor uses similar tasks over time, or collaborates with colleagues, they can develop a catalog of student work samples to guide future teaching. As I described in my reflections, I already have a number of productive ideas for how to modify the lessons and my teaching next time around. The use of the five practices also provides insights into the types of tasks that work well. For instance, I found that iterative sequences of functions were productive areas of inquiry, because they were constructive and allowed for many opportunities for students to discover patterns.

The five practices can be supported with the use of other tools. For instance, carrying around a clipboard during class sessions to write down notes about students is one effective way to monitor, select, and sequence ideas. If these notes are kept over time, they can serve as a reference for the next time a lesson is taught. Indeed, such notes provided the basis for describing student work in this article. In addition, time invested in developing group norms improves the general productivity of students in a team environment. Ultimately, executing the five practices takes practice, but they can considerably improve the quality of classroom discourse. Because they help bring together the ideas of diverse students in a supportive and constructive way in classroom discussion, they have great potential for supporting student learning.

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Figure 1. Student construction of first- and second-order Hilbert space-filling curves.

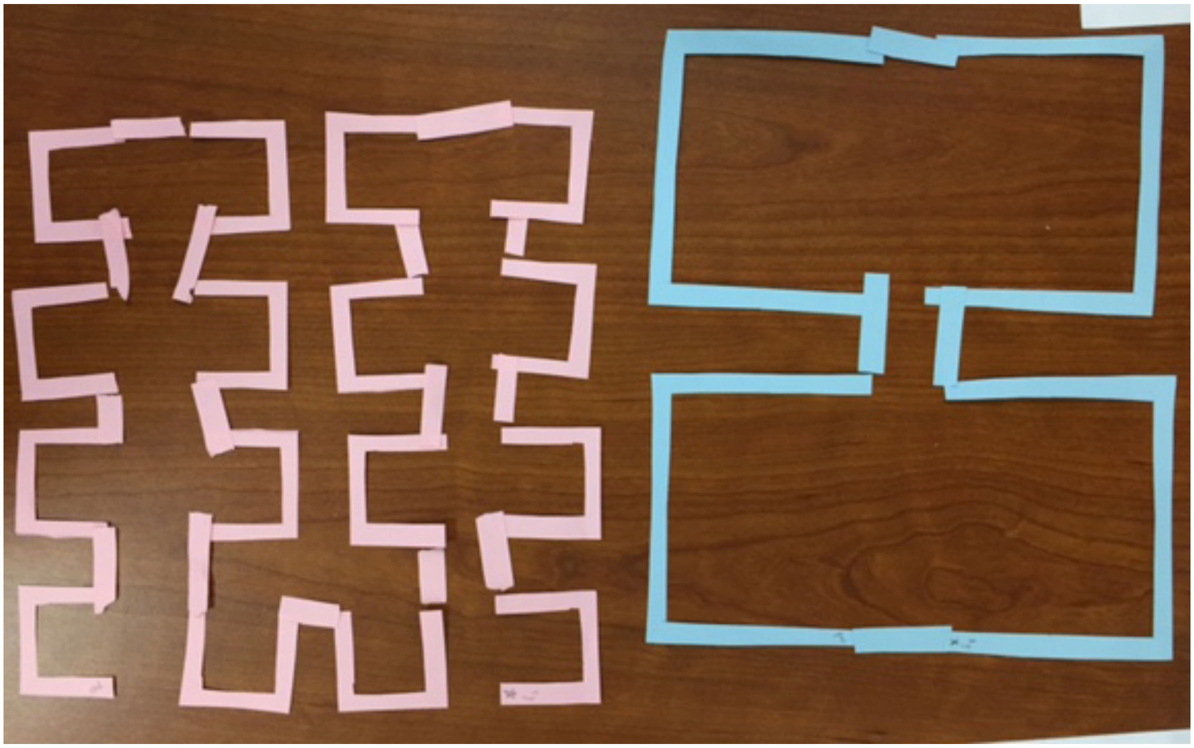

Figure 2. Students constructing Fourier series on the sidewalk.

Figure 3. Two iterations of the Z-order space-filling curve.

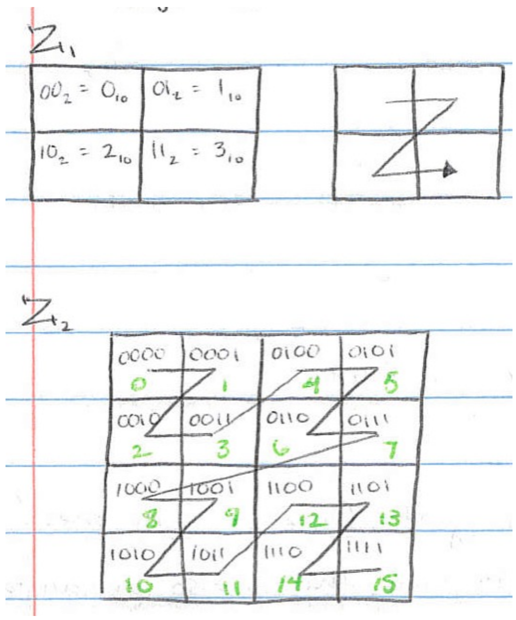

Figure 4. Second-order space-filling curve with compass directions.

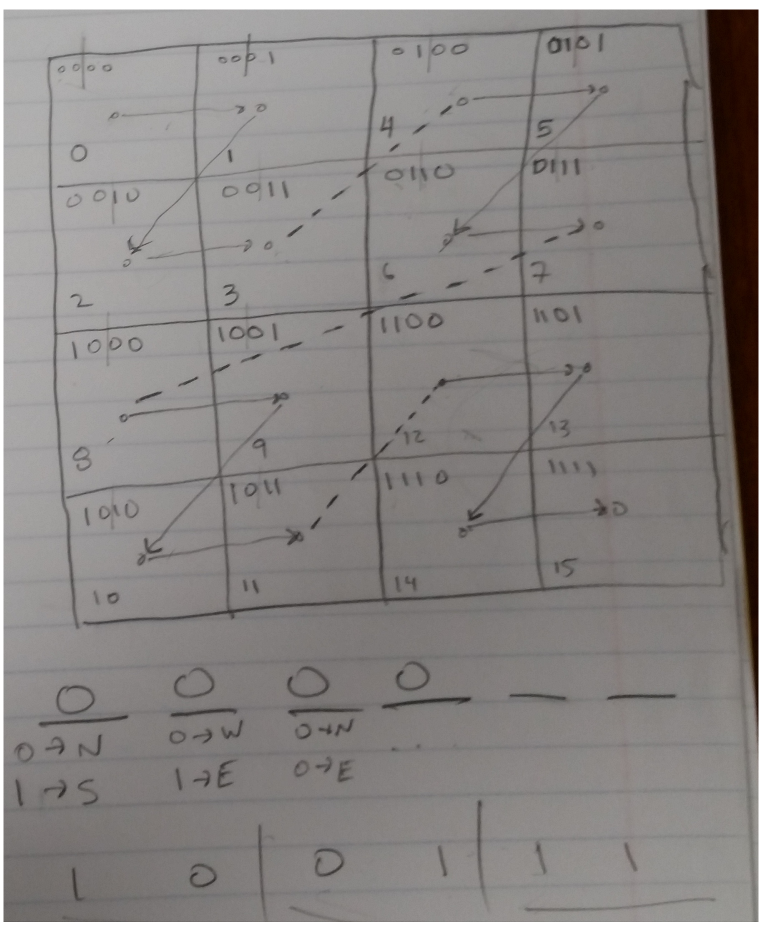

Figure 5. Third-order space-filling curve with compass directions.

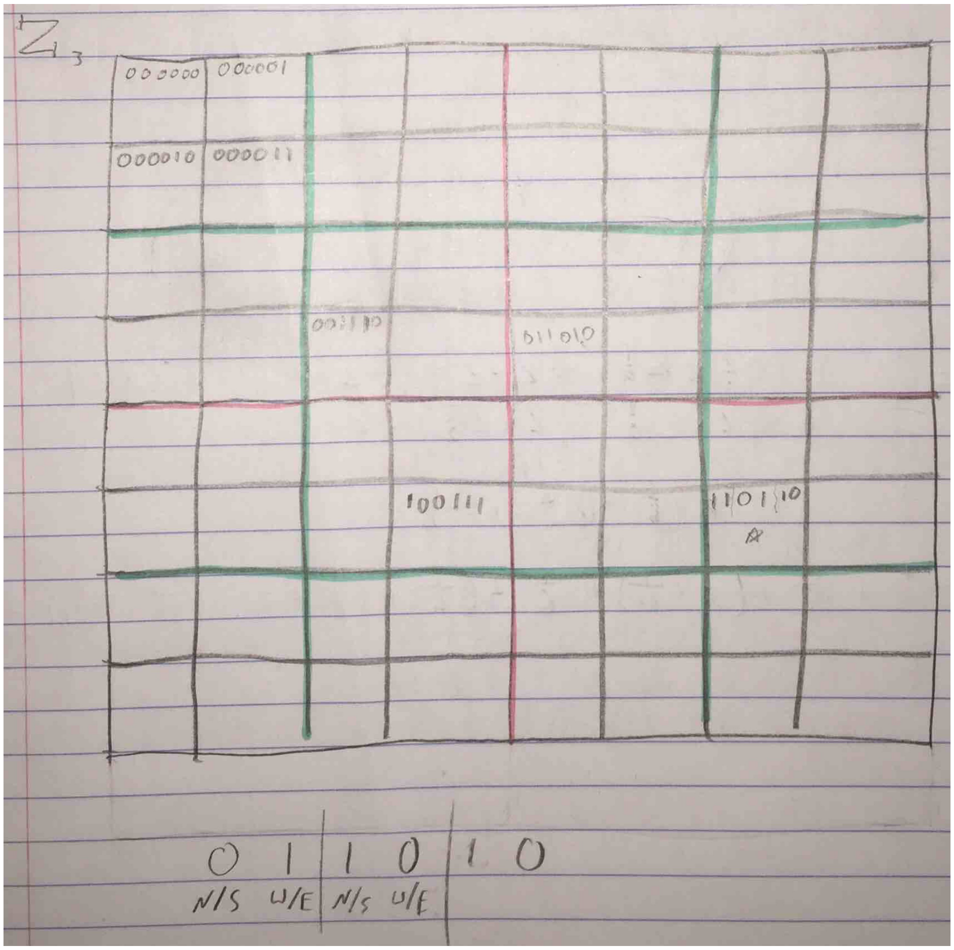

Figure 6. Definition of the Lebesgue function.

The Lebesgue function $L(x)$ is defined as the result of an iterative sequence $\{f_n\}_n$, as follows:

1. Let $f_0(x) = x$.
2. For $n \geq 0$, let $f_{n+1}(x) = \begin{cases} 0.5 f_n(3x) & 0 \leq x < 1/3 \\ 0.5 & 1/3 \leq x \leq 2/3 \\ 0.5 + 0.5 f_n(3x - 2) & 2/3 < x \leq 1 \end{cases}$

Figure 7. Student construction of the Cantor function.

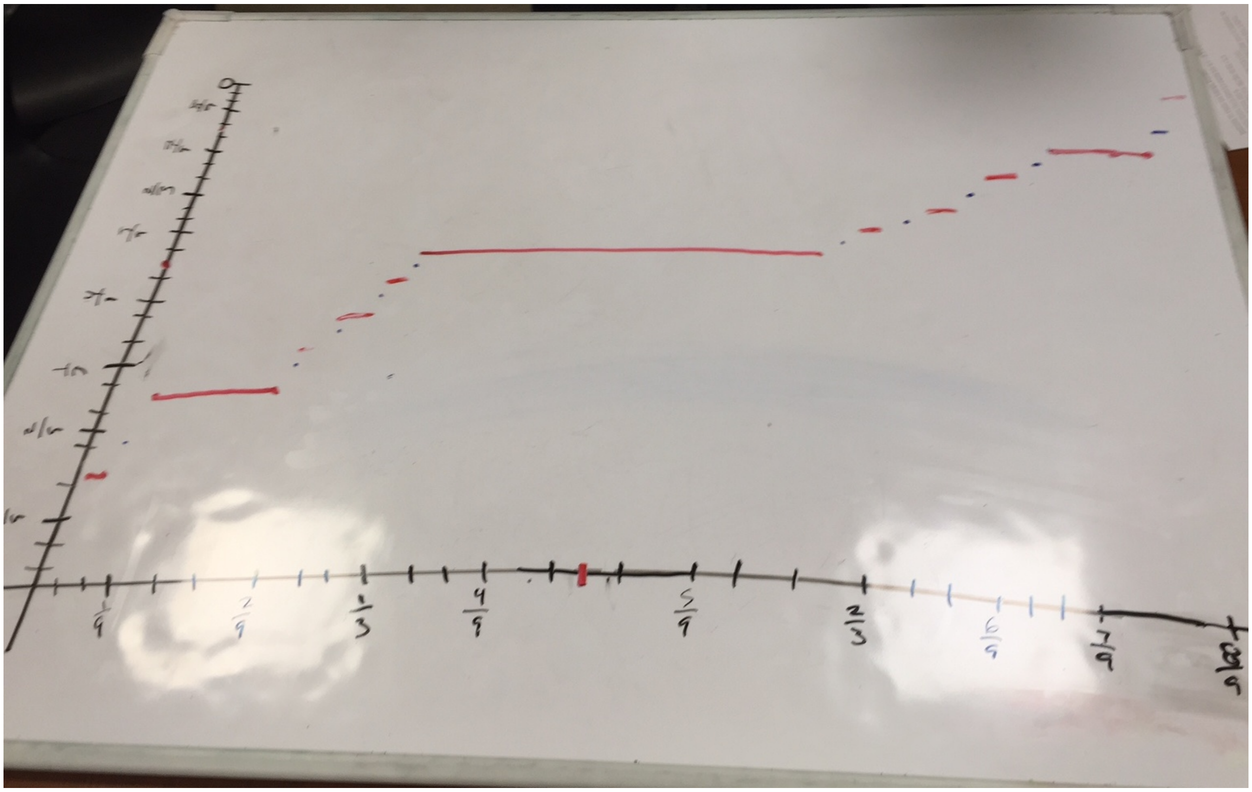

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Figure 8. Student construction of the sequence of Lebesgue functions.