
Peer Feedback for Learning Mathematics

Daniel L. Reinholz

Abstract. This article describes Peer-Assisted Reflection (PAR), a cycle for engaging students with rich mathematical problems. PAR gives students an opportunity to receive peer feedback and revise their work, supporting sustained mathematical engagement. PAR can be incorporated into nearly any course to address logistical constraints that limit how much feedback an instructor can provide. Strategies for using PAR and sample problems are provided.

1. INTRODUCTION. Good problems are like gems. They entice, inspire, and engage us. They are sources of creativity and innovation. They steal our attention: distracting us in hallways, making us late to meetings, or keeping us up at night. Arguably few delight in grappling with problems the way that mathematicians do. How do we help our students develop this same love for problems? Problem solving is not a clandestine tryst; it is a journey that requires sustained engagement. How do we help students develop the perseverance to not simply give up on challenging problems?

This article describes Peer-Assisted Reflection (PAR), a teaching technique designed to address these challenges. PAR gives students an opportunity to receive peer feedback and revise their work on interesting problems. As students engage in iterative problem solving, they are exposed to the importance of working on problems over extended periods of time. With supportive peer feedback, they are able to achieve much more than they would alone. This article describes the theory of PAR and its learning benefits, providing practical guidance on its use.

2. BACKGROUND. Feedback is a key part of learning. However, not all feedback is equally useful [3]. Feedback works best when it is timely, specific, and actionable [11]. In contrast, when students only receive feedback after the fact, it is less likely to support their learning, because they do not actually get to use it. Moreover, there is value in separating feedback from grading. When students receive grades *and* written comments, students focus on the grades and consequently learn less from the feedback [2, 4]. The PAR cycle builds on this research to provide students with useful feedback.

During a weekly PAR cycle, students: (1) complete a draft solution to a homework problem, (2) reflect on their work, (3) exchange peer feedback, and (4) revise before submission. All of these activities take place outside of the classroom, except for the peer feedback exchange, which requires ten minutes of class time. While students have less mathematical expertise than their instructor, the nature of the PAR cycle allows them to give immediate, compelling feedback that their peers can actually use. For instance, students may identify pronouns such as “this,” “that,” or “it,” with ambiguous referents; they may also note logical errors in the use of quantifiers or negating a definition. Then, these small but potentially fatal errors can be corrected before grading.

PAR has been studied in a variety of contexts with positive results. For instance, during two semesters in introductory calculus, students who used PAR had improved passing rates by 13% (first semester) and 23% (second semester) compared to those who did not [7]. These students also learned to communicate mathematics more effectively [9]. PAR helps students learn both through receiving and *giving* feedback [6]. When students give feedback they are exposed to alternative perspectives and learn to think more critically about their own work [1, 10]. Still, student feedback is not always

mathematically correct and students sometimes ignore peer feedback; nevertheless, on average, students improve their work when they revise as a part of PAR [7].

3. USING PAR. PAR is a relatively simple teaching technique that can be added to any course, whether lecture-based or focused on student group work. PAR is not a fixed format, but a starting place for deeper learning. Accordingly, the following suggestions should be modified to fit an instructor's needs and context.

Logistics. A *PAR packet* organizes student work as follows: the problem statement, a feedback and reflection form, the draft solution, and the revised solution (see <https://www.github.com/reinholz/PAR> for a sample PAR packet). The feedback form consists of a set of icons between two columns. To guide peer feedback, students circle icons representing areas they want to focus on and cross out icons they want to ignore. This helps students reflect critically on their work. The columns are for feedback: the right column for strengths and the left column for areas of improvement.

Students complete a first draft and fill out the reflection form before they come to class. Then they conference in class. A PAR conference has five minutes for silent feedback and five minutes for discussion. Requiring silent feedback first ensures that students engage with peer work rather than simply talk about the problem. At the beginning of the semester, students will need to be reminded to first provide silent feedback. In addition, mixing up student partners (e.g., through random assignment) provides students with a range of different perspectives.

PAR assignments are graded both for correctness and for process. An instructor gives 2 points for revision, 2 points for receiving peer feedback, and 6 points for correctness of the submission; the correctness of the initial draft is ignored. Grading for process encourages students to attend class and actually do PAR. If students miss class, they can conference outside of class to receive full credit.

Framing. Students need to understand the purpose of PAR. Instructors should emphasize the role of communication in professional practice (e.g., proof, scientific explanations) and how science is built on peer review. They should also draw attention to the vast literature connecting explanation and learning [5]. One intuitive way to summarize this literature is to describe how teaching is one of the best ways to learn. At first students may express skepticism, because this is an unfamiliar activity, so it is important to encourage them. Over time, students will see the value of the process.

Enhancing Feedback Quality. One potential challenge with PAR is teaching students to provide better feedback. When students make vague statements such as “explain more,” it provides little guidance for their peers on how to improve. Fortunately, students can learn to provide more specific feedback with minimal support [8]. One effective method is for an instructor to bring in sample student work (real or hypothetical) that shows how students worked on a particular problem. Then, the class discusses the strengths and weaknesses of the work and how to provide feedback to this hypothetical peer. Include such activities on a semi-regular basis (about once a week) during the first half of the semester. While this requires some time, it provides a meaningful opportunity for discussing areas of student struggle, so in reality, class time is not lost.

Problem Choice. Good PAR problems: (1) are easy to start but hard to master, (2) afford multiple solution paths, and (3) are more than just computations. Such problems result in better conferences, because students can build on each other's partial understanding and provide alternative viewpoints. In contrast, overly difficult or procedural problems leave little room for discussion.

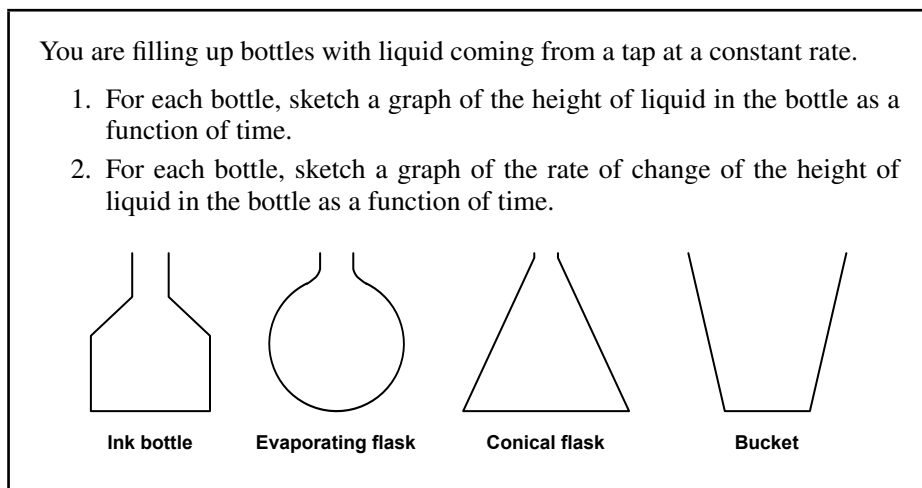


Figure 1. Filling bottles PAR problem.

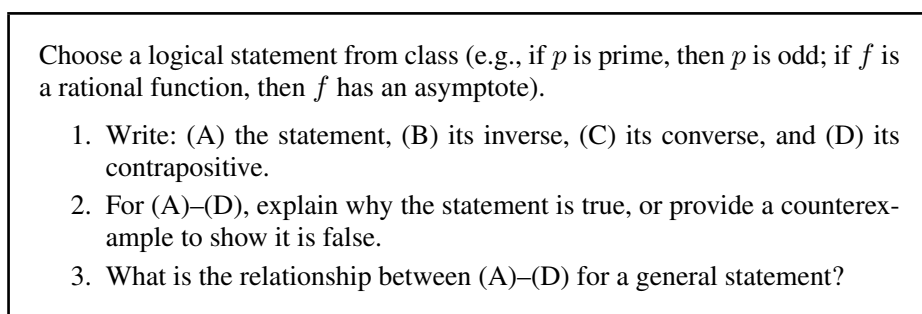


Figure 2. Logic PAR problem.

A few example problems are given. Graphical reasoning problems (e.g., Figure 1, from calculus) can lead to productive conversations about mathematical representations. True/False questions (e.g., Figure 2, from introductory logic) require students to explain their thinking, and students almost always come up with different counterexamples. Open-ended problems with opportunities to choose different paths allow for students to compare their work (e.g., Figure 3, from analysis).

4. AN ILLUSTRATION. An illustration of PAR is now provided. Figure 4 provides a draft solution for the filling bottles problem (see Figure 1). This was a reasonable attempt, but lacked some clarity in the graphs. During the peer conference, a student provided the following feedback:

- Describe which flask features would cause the graph to be increasing/decreasing.
- Identify the points of change in the graph.

After receiving this feedback and having a conference, the student revised their

We constructed Cantor's middle thirds set by progressively removing the middle third from an interval (starting with $[0, 1]$). We can create other Cantor sets by removing different amounts.

1. Choose the width of the interval you will remove (e.g., $8/10$, $1/4$). Draw three iterations of your set.
2. How much "space" gets removed during each iteration?
3. Can you find numbers that are not eventually removed?
4. In general, what do the numbers in your set look like?

Figure 3. Cantor set PAR problem.

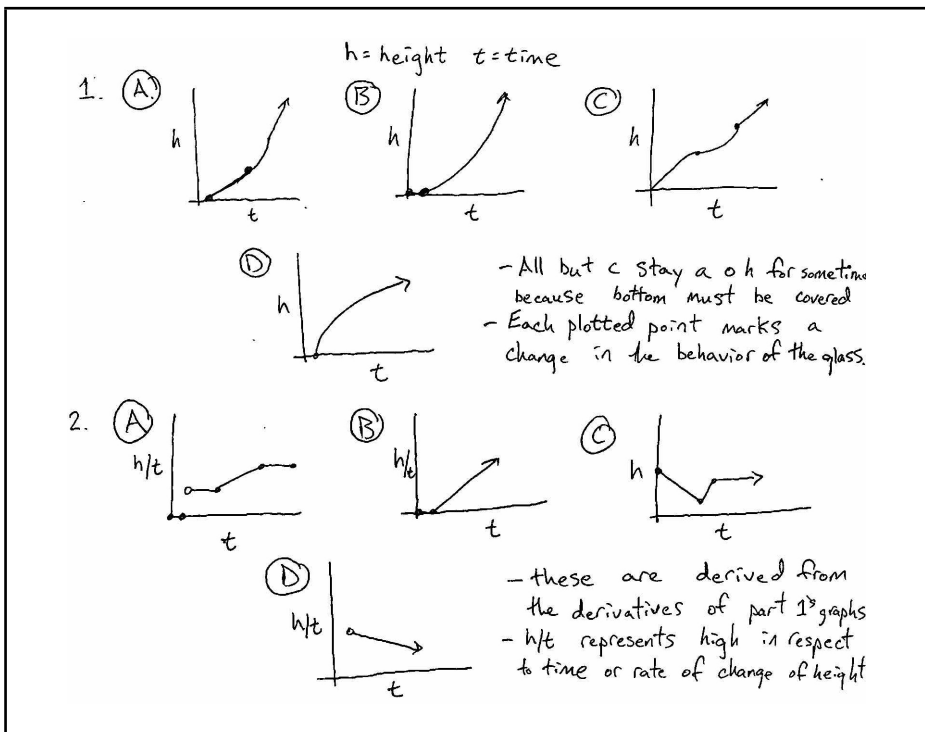


Figure 4. Student draft solution.

work as in shown in Figure 5. Here the student more clearly marked the transition points on each of the graphs and provided more detailed explanations of the shapes. This does not mean that the submitted solution was completely correct, but it was still greatly improved. Student improvement after revision is typical.

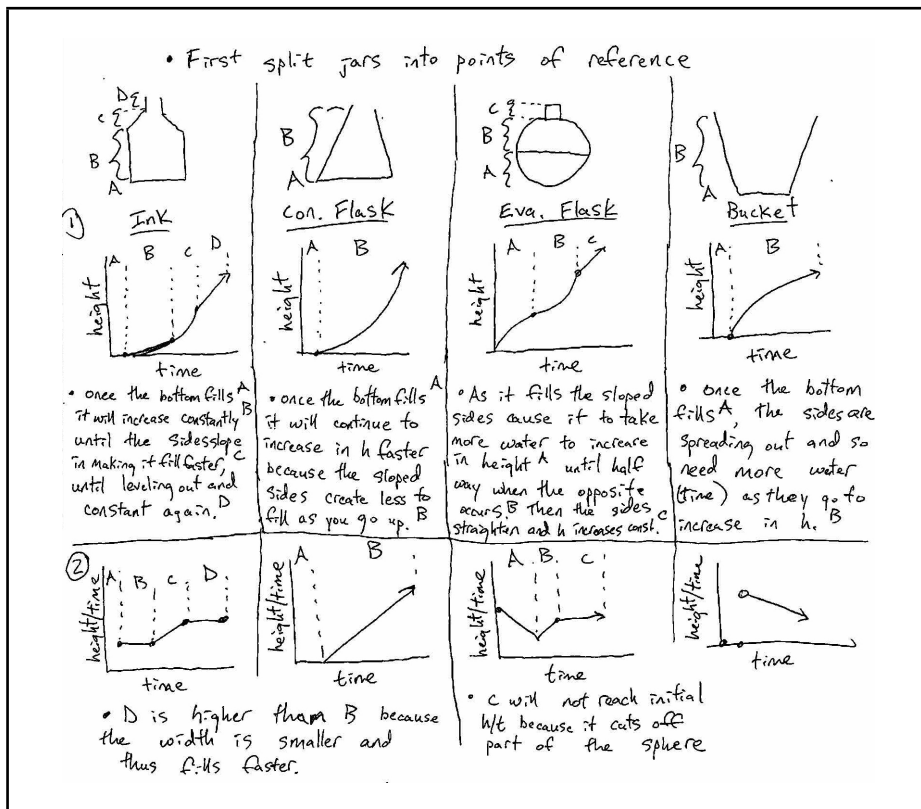


Figure 5. Student submitted solution.

5. PAR FOR PROOF. Although PAR was developed for calculus, it has since been adapted to a variety of contexts, including: differential equations, analysis, introductory mechanics, thermodynamics, and genetics. For problem-solving courses, one may use PAR as described above. For proof-based courses, some modifications are required. First, rather than having a separate feedback form, students complete proofs in a two-column format, where the left column is for the proof and the right column is for annotations and feedback. This also allows students to annotate their own proofs to communicate their thought processes to their peers, and peers annotate in pen to distinguish their comments. Discussions about proofs may also take more class time, so it helps to extend beyond 10 minutes to 15 or 20 minutes. Finally, students annotate their final submission to “respond to reviewer comments,” explaining how they did or did not use the feedback they received.

6. DISCUSSION. Sustained engagement with challenging problems can be supported with useful feedback and opportunities to revise. PAR has been used productively across settings to provide such feedback. Although students have less mathematical sophistication than an instructor giving feedback, PAR has added benefits: (1) it decouples feedback from grading, (2) students learn from giving feedback, and (3) students have time to discuss their feedback verbally. Because PAR is primarily focused on homework, it can be incorporated into nearly any classroom teaching style

with minimal effort.

References

- [1] P. Black, C. Harrison, C. Lee, *Assessment for learning: Putting it into practice*, Open Univ. Press, Berkshire, England, 2003.
- [2] R. Butler, *Enhancing and undermining intrinsic motivation: The effects of task-involving and ego-involving evaluation on interest and performance*, *British Journal of Educational Psychology* **58** no. 1 (1988) 1–14.
- [3] J. Hattie, H. Timperley, *The power of feedback*, *Review of educational research* **77** no. 1 (2007) 81–112.
- [4] A. A. Lipnevich, J. K. Smith, *Response to Assessment Feedback: The Effects of Grades, Praise, and Source of Information*, ETS Research Report Series **2008** no 1. (June 2008) i–57.
- [5] T. Lombrozo, *The structure and function of explanations*, *Trends in cognitive sciences* **10** no 10. (2006) 464–470, <https://doi.org/10.1016/j.tics.2006.08.004>.
- [6] D. L. Reinholz, *The assessment cycle: A model for learning through peer assessment*, *Assessment & Evaluation in Higher Education* **41** no. 2 (2016) 301–315, <http://doi.org/10.1080/02602938.2015.1008982>.
- [7] ———, *Peer-Assisted Reflection: A design-based intervention for improving success in calculus*, *International Journal of Research in Undergraduate Mathematics Education* **1** no. 2 (2015) 234–267, <http://doi.org/10.1007/s40753-015-0005-y>.
- [8] ———, *Peer conferences in calculus: The impact of systematic training*, *Assessment & Evaluation in Higher Education* **42** no. 1 (2017), 1–17, <http://dx.doi.org/10.1080/02602938.2015.1077197>.
- [9] ———, *Improving calculus explanations through peer review*, *The Journal of Mathematical Behavior* **44** (2016), 34–49, <http://doi.org/10.1016/j.jmathb.2016.10.001>.
- [10] D. R. Sadler, *Formative assessment and the design of instructional systems*, *Instructional science* **18** no. 2 (1989) 119–144.
- [11] V. J. Shute, *Focus on formative feedback*, *Review of Educational Research* **78** no. 1 (2008) 153–189.

Department of Mathematics and Statistics, San Diego State University, San Diego, CA 92182
daniel.reinholz@sdsu.edu