Developing mathematical practices through reflection cycles

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This paper focuses on reflection in learning mathematical practices. While there is a long history of research on reflection in mathematics, it has focused primarily on the development of conceptual understanding. Building on notion of learning as participation in social practices, this paper broadens the theory of reflection in mathematics learning. To do so, it introduces the concept of reflection cycles. Each cycle begins with *prospective* reflection, which guides one's actions during an experience, and ends with *retrospective* reflection, which consolidates the experience and informs the next reflection cycle. Using reflection cycles as an organizing framework, this paper synthesizes the literature on reflective practices at a variety of levels: (1) metacognition, (2) self-assessment, (3) noticing, and (4) lifelong learning. These practices represent a spectrum of reflection, ranging from the micro level (1) to macro level (4).

Keywords: reflection, metacognition, self-assessment, teacher noticing, self-regulation

Introduction

Reflection is a fundamental part of learning. In mathematics, reflection has been considered primarily as a tool to promote conceptual understanding (e.g., Cobb, Boufi, McClain, & Whitenack, 1997; Simon, Tzur, Heinz, & Kinzel, 2004), or the *acquisition* of concepts (Sfard, 1998). Yet, the sociocultural turn (Lave, 1996; Vygotsky, 1978) also emphasizes *participation* in practices. These two metaphors for learning – acquisition and participation – are both useful; neither alone is sufficient (Sfard, 1998). This theoretical paper aims to broaden the scope of reflection research in mathematics education to emphasize practices in addition to concepts. There are two primary arguments: (1) reflection can promote the development of practices, and (2) reflection structured after a learning process (retrospective) can be used to guide future learning processes (prospective).

Mathematical practices feature prominently in standards documents across the world. For instance, the National Statement on Mathematics for Australian Schools states that “Students should experience the processes through which mathematics develops” (Australian Education Council, 1990, pp. 22–23); mathematics is something that one does, not just something one knows. Building on this, the Australian Curriculum states: “a fundamental aim of the mathematics curriculum is to educate students to be active, thinking citizens, interpreting the world mathematically, and using mathematics to help form their predictions and decisions about personal and financial priorities” (ACARA, 2009). The Danish KOM project to define mathematical competence similarly concluded: “Possessing mathematical competence means having knowledge of, understanding, doing and using mathematics…” (Niss, 2011, p. 17). Consistent with this, the Common Core State Standards for Mathematics explicitly call out two strands – practices and content – as crucial to mastery of mathematics. Given the value of reflection for learning and the call of the above standards documents, the question arises: what role might reflection play in the development of mathematical practices?

While researchers outside of mathematics education generally agree that reflection is a tool for guiding future actions (Pavlovich, 2007), different types of reflection may support future actions in different ways. Most commonly, reflection is used *retrospectively*, as a means of processing and better understanding a particular lived experience after it takes place (Kennison & Misselwitz, 2002). By processing an experience in depth, an individual develops a deeper understanding that ideally supports better-grounded future actions. Reflection may also be used *prospectively,* as a lens for guiding one's observations and noticing before and during an experience. In this way, reflection shapes the actual experience itself, by opening up space to respond to events as they unfold. These types of reflection are distinct but related (Boud & Walker, 1991).

These two types of reflection can be understood in terms of reflection cycles. Reflection cycles draw from Schön's (1983) notion of *reflection-in-action* and Kolb's (1984) model for experiential learning. Each reflection cycle represents an experience in mathematics teaching or learning. An individual's actions during a reflection cycle are guided by prospective reflection, and at the end of each cycle, learning is consolidated through retrospective reflection. As an individual engages in multiple reflection cycles over time, the interplay between prospective and retrospective reflection becomes evident. The concept of reflection cycles provides a framework for understanding reflection over different timescales. As such, reflection cycles help organize the vast literature on reflection. This particular paper considers reflection that occurs at four levels (from micro to macro): metacognition, self-assessment, noticing, and lifelong learning. Examples from undergraduate mathematics are provided at each level.

Background: Reflection for Conceptual Development

The study of reflection in mathematics education has focused primarily on conceptual understanding, drawing mostly from Piaget's genetic epistemology, which describes the development of cognitive structures (Piaget, 1972). Cognitive structures develop primarily through two related processes: *assimilation* and *accommodation*. Consider a basic sensorimotor scheme, swatting a fly (Piaget, 2001). When a child encounters a hornet, it may *assimilate* the environment to the scheme, applying it without any modifications. However, the scheme may not result in the desired effect (squashing the fly), instead resulting in the child getting stung. In this case, the child needs to *accommodate* the scheme to the environment, modifying it in the process (e.g., by limiting its domain of application to not include hornets). To explain the development of more abstract knowledge structures, those not directly grounded in sensorimotor experience, Piaget introduced the idea of reflective abstraction.

Reflective abstraction involves the *projection* of lower-level cognitive structures to a higher level and the reorganization or *reflection* of these structures to integrate them with other higher-level cognitive structures (Piaget, 2001). Thus, reflective abstraction allows new cognitive structures to be built out of existing structures. Consider the example of multiplication. Although multiplication can be thought of as "equivalent" to repeated addition (at least for natural numbers), it is often more difficult for students to learn than addition. According to Piaget's theory, one reason is that understanding multiplication requires that: (1) students recognize how much they are adding each time and that (2) they recognize the number of times they are adding the same amount. Most children have little problem with the first process, which is grounded in perceptual experience. However, the second process requires abstracting the qualities of one's own actions (i.e. reflective abstraction), which can be more difficult.

The above concepts underlie a large body of mathematics education research. As Gray and Tall (1994) note, "the notion of actions or processes becoming conceived as mental objects has featured continually in the literature" (p. 118). This can be seen in accounts of conceptual development through reflective abstraction (Simon et al., 2004), reflective discourse (Cobb et al., 1997; Tanner & Jones, 2000), reification (Sfard, 1991) and Action-Process-Object-Schema (APOS) theory (Dubinsky & McDonald, 2002). Each of these theories focuses on how a mathematical action becomes a mathematical object in its own right. Through this type of reflection, a thing that is *done* becomes a thing that *is.* Creating such mathematical objects is an important part of developing deeper conceptual understanding. For instance, a mathematical function can be considered as a *process* for assigning a set of inputs to a set of outputs, or this relationship can be thought of as an *object*, allowing for the function itself to be acted upon (e.g., by mathematical operators). This duality of process and object is one of the reasons that learning functions can be so difficult (Dubinsky & Wilson, 2013). In the case of reflective discourse, one is interested in how discussion at the group level supports the development of mathematical concepts at the individual level, such as the "flexible partitioning" of numbers (Cobb et al., 1997). As these studies highlight, reflection plays an important role in the development of mathematical concepts.

Theoretical Framing: Reflection on Practices

Although most research in mathematics education has focused on conceptual development, some researchers have considered other notions of reflection, such as in the study of dialogue and critical mathematics (Alrø & Skovsmose, 2003) and teacher education (Averill, Drake, Anderson, & Anthony, 2016). Outside of mathematics, reflection takes on an even larger variety of meanings (cf. Atkins & Murphy, 1993; Thorpe, 2004). As a result, it has been difficult to generalize the results of research on reflection (cf. Gore & Zeichner, 1991; Mackintosh, 1998; Mann, Gordon, & MacLeod, 2009).

Despite conflicting definitions of reflection, many studies highlight the relationship between reflection and experience. Accordingly, this paper defines reflection as follows: *reflection is the act of processing an experience, action, or practice.* Reflection often involves thinking about one's thoughts, feelings, or actions related to an experience. Through reflection, an individual attempts to gain further insight into the experience to better inform and guide action (Pavlovich, 2007).

Learning through reflection traces back to Dewey (cf. Lyons, 2010). Dewey (1933) believed that the purpose of reflective thought was to transform a situation "in which there is experienced obscurity, doubt, conflict, disturbance of some sort, into a situation that is clear, coherent, settled, harmonious" (p. 100). Dewey emphasized that reflective thinking arises from situations, and that it is experience itself that calls out for reflection; reflection is not strictly internal processing, but rather a response to the environment. Dewey's work is foundational to experiential learning, as in Kolb's (1984) model of the experiential learning cycle consisting of four steps: concrete experience, reflective observation, abstract conceptualization, and active experimentation; reflection is a means of processing experience to make abstractions. Boud, Keogh, and Walker (1996) also claim "[r]eflection is a form of response of the learner to experience." Boyd and Fales (1983) similarly state: "[r]eflective learning is the process of internally examining and exploring an issue of concern, triggered by an experience…" These definitions highlight that reflection is more than "thinking deeply" (Wilson & Clarke, 2004), which may not relate to processing experience at all. The connection to processing experiences is what makes reflection a tool for developing practices.

Reflection research has surged since Schön's (1983) description of professionals engaging in *reflection-in-action.* The crux of Schön's argument is that individuals (e.g., designers) possess a great deal of tacit knowledge that they can only access by actually doing something (Schön, 1992). As such, design can be characterized as a "reflective conversation with the situation" (p. 4), in which a designer tests conjectures about how to design something by actually designing it. Each time the designer enacts a design decision she reflects upon it, which provides information to guide the next decision, and so on. Similarly, other professionals (e.g., psychologists), develop conjectures about what is happening in a given situation (e.g., why a patient is experiencing certain emotions) that they then enact so that they can reflect upon them (e.g., by having the patient try a certain course of treatment and seeing how it goes). The knowledge required to reflect-in-action is taken as something that experts have developed over time. A key takeaway from Schön's work is that reflection not only takes place after the fact, but also in the moment.

Building on this distinction, I introduce the concepts of *retrospective reflection* and *prospective reflection*. *Retrospective* reflection, which occurs after the fact, helps one understand an experience that has already taken place. This is the most common type of reflection studied in the literature, such as in the case of reflective journals (Moon, 1999). In contrast, *prospective* reflection guides one's actions during an experience, a hallmark of competent disciplinary and professional performance (Hatton & Smith, 1995). For instance, in mathematics, this anticipatory ability helps a problem solver notice and select which features of a problem are relevant and how to engage with them (Niss, 2010). By developing the ability to reflect prospectively, individuals can learn to guide their actions in new ways.

While similar to reflection-in-action, prospective reflection is more general. The key idea with reflection-in-action is that individuals *act* in a given situation so that they can *reflect* on the impact of their actions. In contrast, prospective reflection encapsulates what one notices and attends to as an event unfolds, and how these noticings are used to guide one's actions (Sherin, Jacobs, & Philipp, 2011). Thus, while reflection-in-action is a form of prospective reflection, an individual need not actually modify a situation to engage in prospective reflection, which makes prospective reflection a broader concept.

A major part of prospective reflection is learning what to attend to and how to attend to it. This is context-specific and value-laden; in this way, learning to reflect prospectively is an important component of being enculturated into a particular community of practice (Lave, 1996). When retrospective and prospective reflection are organized together, the purpose of the retrospective reflection is to help an individual learn what to attend to. Over time, this shift in focus of attention becomes more automatic, and the individual learns to attend to these relevant features *during* an experience (prospectively) rather than only after the fact. This is key, because learning to change one's behaviour in the moment, rather than after the fact, allows for the development of new practices.

A reflection cycle is defined as the unit of experience over which reflective practices take place. These units may occur during micro interactions in problem solving (e.g., in the case of metacognition), or over long macro periods of learning through which individuals restructure their global learning processes (e.g., in the case of developing lifelong learning skills). In either case, an individual's actions during a given cycle are guided by prospective reflection, and at the end of a cycle, learning is consolidated through retrospective reflection. As an individual engages in similar experiences over time, retrospective reflection from one cycle can influence how prospective reflection takes place in subsequent cycles. However, if an individual does not reflect retrospectively after a given cycle, it is less likely that their prospective reflection in future cycles will shift considerably. In this way, it is the interaction between these two types of reflection that supports the change of an individual's practices over time.

Reflection at Multiple Levels

Reflection describes a wide variety of activities that involve processing experience to inform future actions, each of which has its own research literature. This paper focuses on four categories of activities, which are related to: (1) metacognition, (2) self-assessment, (3) noticing, and (4) lifelong learning. Not intended to be exhaustive, this synthesis of literature provides insight into how the concept of reflection cycles can be used to understand reflection over various timescales.

Metacognition relates to the monitoring and control that takes place moment-to-moment as an individual engages in a practice such as problem solving (Schoenfeld, 1987). As such, a single problem-solving episode may consist of numerous reflection cycles through which an individual modifies their reflective practices. At the next level, self-assessment refers to how an individual makes sense of what they do or not know in a given problem context (Reinholz, 2015b). This is generally, but not necessarily, a longer timescale than metacognition; prospective reflection guides self-assessment during a single problem, and retrospective reflection consolidates this learning at the end of a problem. Noticing occurs at a longer timescale, related to how a teacher learns to attend to and respond to different aspects of classroom practice (Sherin et al., 2011). Reflection also occurs more globally in developing lifelong learning skills, as in a learner modifying study habits over months or years (Zimmerman, 2002).

Each of these activities or practices is related to mathematics and teaching and learning, yet they all occur at different scales. As such, their underlying unity as a set of reflective practices is generally overlooked. From micro to macro, prospective reflection influences: (1) the selection of strategies and decisions that guide the problem-solving process (metacognition), (2) the features in a solution that an individual focuses on (self-assessment), (3) what is noticed in the classroom and how one responds to it (teacher noticing), and (4) the global learning strategies, such as study skills, that guide the learning process (lifelong learning). In what follows, the relation between these practices and reflection is elaborated. Four examples from undergraduate mathematics are given. In the analysis of these studies the language of prospective and retrospective reflection is used, to show how the framework of reflection cycles applies.

Metacognition

Metacognition traces back to Flavell (1979), who defined it as: "one's knowledge concerning one's own cognitive processes and products or anything related to them" (p. 232). Brown (1987) built on and expanded this definition, focusing on other aspects of metacognition such as planning, checking, monitoring, and self-assessment. Specific to mathematics, metacognition often relates to Polya's work on problem solving (Polya, 1945). Both Schoenfeld (1985) and Garofalo and Lester (1985) related metacognitive monitoring and strategic problem-solving behaviour. Schoenfeld (1987, pg. 190-191) defines this aspect of metacognition as:

> Control, or self-regulation…Aspects of management include (a) making sure you understand what a problem is all about before you hastily attempt a solution; (b) planning; (c) monitoring, or keeping tack of how well things are going during a solution; and (d) allocating resources, or deciding what to do, and for how long, as you work on the problem.

In this sense, metacognition is concerned with one's awareness of their problem solving processes and their ability to use that awareness to guide problem solving (cf. Goos, 1994). Metacognition is a reflective practice, as it relates to how one *processes their experience* of problem solving.

Reflective practice is characteristic of expert problem solving. Consider the following comparison of talented undergraduates and professional mathematicians

(Schoenfeld, 1985). When faced with a nontrivial, unfamiliar problem, the undergraduates generally engaged in read-explore behaviour: they would read the problem, decide on a course of action, and continue along that path. Students often continued on a single path for 20 minutes (the time allotted to them), even though they were clearly not making progress, and as a result, they would not complete the problem. Schoenfeld described these episodes as "wild goose chases," in which the students did not reflect on what they were doing or try to modify their plan of action (Schoenfeld, 1987). In contrast, the professional mathematicians spent time jumping between modes: they might read the problem, plan, explore a little bit, analyse the situation and change course, and cycle through some variation of these behaviours multiple times before actually reaching a solution. The professionals were much more aware of what they were doing and were able to reflect on their processes and change courses multiple times. Given their ability to reflect *prospectively*, the expert mathematicians were able to adapt their behaviour flexibly.

To help students learn to reflect *prospectively*, Schoenfeld developed a problem solving course (Schoenfeld, 1985). Most of the activities employed can be understood through the lens of retrospective reflection: (1) videotapes, (2) teacher role modelling, (3) whole-class discussions, and (4) group problem solving (Schoenfeld, 1987). To begin the course, Schoenfeld showed videotapes of other students solving problems to illustrate the wild goose chases that resulted when students did not reflect on their problem-solving process. Second, when presenting problems at the board, Schoenfeld modelled the messiness of problem solving, highlighting the reflective processes in his own thinking, rather than just showing a cleanly worked out solution. Third, when students discussed problems as a class, he played the role of "metacognitive monitor," helping the class reflect collectively (Goos, Galbraith, & Renshaw, 2002). Finally, when students worked in small groups, Schoenfeld periodically interrupted students to ask one of three probing questions: what are you doing?; why are you doing it?; and where will it get you?.

The activities described above are examples of retrospective reflection being used to help students learn what to attend to in problem solving. In particular, these reflective activities made explicit the need to plan, monitor, and control one's behaviour. Over time, students learned that these were key components of problem solving, and were able to use them *prospectively* to guide their problem-solving activities. In other words, the reflective activities that were used retrospectively at the end of each reflection cycle (which consisted of problem solving) supported prospective reflection in future cycles. For example, Schoenfeld described student responses to his "three questions" as follows (Schoenfeld, 1987, pp. 206–207):

> At first, the reaction from each small group is an embarrassed silence…Soon the students realize that I'm serious about the questions and that I will continue to ask them even though doing so makes them feel uncomfortable. To defend themselves against these intrusions, they begin to prepare answers to the questions in advance. Over the course of the semester, the students get in the habit of discussing the questions, both at the beginning of the problem sessions and at major decision points during problem solutions. When things work well, discussions of the underlying issues…become a matter of practice.

What Schoenfeld describes is precisely the process of students moving from *retrospective* reflection, through external prompting, to *prospective* reflection, in which reflective thinking became a tool used to guide their problem solving. Upon finishing his class, only 20% of the problem solving attempts made by students were of the read-explore type, compared to 60% of student attempts before taking the course (Schoenfeld, 1987).

Self-Assessment

Self-assessment relates to how well an individual can assess whether or not they know something. A wide variety of studies show that self-assessment is difficult, and that individuals generally struggle self-assess accurately (Dunning, Heath, & Suls, 2004; Dunning, Johnson, Ehrlinger, & Kruger, 2003). Despite their struggles to self-assess, individuals are relatively successful at assessing the work of others. This is because peer assessment takes place from a distanced perspective, which makes it easier to see gradations in quality. In contrast, individuals are generally too close to the specifics of their own work to see its flaws (Black, Harrison, & Lee, 2003). For example, individuals generally understand the logic of their writing, storytelling, or mathematics solutions, so it is easy for them to fill in missing or unclear details without even recognizing that their work is incomplete. A secondary reader, however, does not have the benefit of this additional knowledge, so the gaps in the communication are easier to see.

Peer-Assisted Reflection (PAR) is a learning activity that leverages peer assessment to support self-assessment (Reinholz, 2015b). The PAR process consists of four steps through which students: (1) work on a difficult problem, (2) self-assess, (3) assess peer work and exchange feedback, and (4) revise before turning in a final solution. Steps (1), (2), and (4) all occur outside of class. The third step, peer assessment and feedback, occurs in class; students are given approximately five minutes to given written feedback (silently), and five minutes to discuss their feedback with one another.

By having students reflect retrospectively on the quality of their solutions and their peers' solutions, PAR helps students develop a better sense of "what counts" for a high-quality solution. Ultimately, students learn to use this sense prospectively to guide their work on future problems. Like metacognition, self-assessment is a reflective practice focused on processing one's problem solving, but it is focused on the output of problem-solving (i.e. the solution) rather than the work of finding the solution.

To help introductory calculus students learn to reflect on their solutions, PAR was introduced in conjunction with a number of supporting activities. First, after completing their draft solution to a PAR problem, students answered a number of reflection prompts (e.g., Did you explain why, not just what? Did you avoid the use of pronouns?; see Reinholz, 2015a). Second, students regularly exchanged feedback with their peers. Because students were positioned as competent and given the charge to make sense of the quality of their peers' solutions, students had to reflect upon what they understood as a high-quality solution. Third, whole-class discussions about the quality of sample student worked were used to make ideas about "high quality" explicit.

The above activities are examples of retrospective reflection. By regularly reflecting, students developed a new sense of high-quality solutions that they could later use to guide how they constructed their own solutions to problems, *prospectively*. In their interviews (Reinholz, 2015a), students described this connection. For instance, Maria described how PAR helped her learn,

> how to make [the solution] easier to read from another person's perspective. It's one thing if I think it looks good, but other people look at it and say it doesn't make sense to me. So [PAR] helps me figure out how to communicate better. It helps me to explain things in a way that is readable to others and not just myself.

In Maria's case, the *retrospective* reflective process of having other students respond to her explanations brought her attention to how she was communicating and how she could communicate differently in the future. This allowed her to reflect *prospectively* as she crafted future solutions. Another student, Harry, noted:

> I really like looking at other people's initial models. I can see what they are thinking, it puts me in their head…

Here, seeing how other students organized their solutions (*retrospective*) gave Harry a new way of thinking as he moved forward in his work (*prospective*). During two semesters of study in introductory calculus, PAR improved students' success; students who engaged in PAR improved their success rates (passing the course with an A, B, or C) by 13% (first iteration) and 23% (second iteration), as compared to students in the comparison sections (Reinholz, 2015a).

Noticing

The emergent literature on teacher noticing provides new frameworks for understanding teacher learning (Sherin et al., 2011). Noticing is a reflective practice, which focuses on how teachers process their classroom experiences. Noticing is often conceptualized as a set of three interrelated skills: (1) *attending* to student thinking, (2) *interpreting* the thinking, and (3) *responding* to that thinking (Jacobs, Lamb, & Philipp, 2010). Through reflective cycles, the types of things that teachers notice, how they interpret them, and what they do with these interpretations all shift.

In general, new teachers are focused primarily on themselves as instructors. Thus, a key goal of professional development programs is to help teachers shift from a teacher-centred to student-centred perspective (Fennema, Franke, Carpenter, & Carey, 1993). Helping teachers make this shift involves influencing what they attend to in the classroom. By focusing teachers' attention to student behaviours, and having them regularly reflect on them, teachers can learn to shift their focus.

This approach was used to help mathematics Graduate Student Instructors (GSIs) learn to modify their teaching practices (Reinholz, Cox, & Croke, 2015). The professional development activities took place during two semesters, in which a total of eight GSIs received support. During any given semester the GSIs met biweekly, for a total of one hour every other week. The goal of the professional development was to help the GSIs reflect on their teaching and help them shift towards a student-centred approach. A number of activities were enacted to support this shift. GSIs: (1) practiced student-centred teaching techniques, (2) had reflective conversations on practice, and (3) conducted peer observations.

During each meeting, the GSIs were given a practice to enact that week during their teaching sessions. This could be using a "turn and talk" or a specific type of deep questioning. During the following meeting, the GSIs had a whole group discussion about their experiences using the practices. In these conversations, the facilitator drew attention to the impact on students, not just what the teachers were doing. Finally, during the second semester, the GSIs engaged in regular peer observations. The peer observation forms were designed specifically to focus on what students were doing (e.g., were they engaged?, who talked to who during groupwork?). In sum, these reflective activities were aimed to focus the attention of the GSIs on students. The goal was that over time the GSIs would attend more to students on their own *prospectively*, and this would be used to guide their attention in teaching.

How this supported learning is highlighted by Beth's story; Beth showed considerable changes in her teaching practices during the first semester (Reinholz et al., 2015). At the beginning of the semester, Beth used some questions during classroom discussions, but rarely pushed students to elaborate their ideas. Over time, she shifted her practices to use probing questions to help students elaborate their ideas and used linking questions to support student-student discussions. She described this as follows (Reinholz et al., 2015, p. 5):

> [I]n the past as a teacher, I would listen to someone, and say I think I know what they are saying and I think they are right but I'm going to rephrase it, like, the right way…I think [this semester] I pushed students more to explain their ideas, and to explain them to each other.

Here Beth describes that initially she was focused on making sense of student ideas for herself, and that *she* would provide a "correct" explanation for the students; this indicates a teacher-centred approach. As the semester continued on, she shifted her practice to focus on having the students explain to each other. In summarizing her growth, Beth described the following (Reinholz et al., 2015, p. 5):

> I've definitely become a lot more reflective about my teaching, and I think about it a lot more than in the past. Just, I question, is this working. I feel like honestly when I first starting teaching, I would just do stuff, and think it's good, oh it's great, and I never really asked, is this working, do I think they are learning this way.

As Beth describes, she learned to pay attention to whether or not students were learning as a result of what she did. This indicates a shift in her focus, towards the impact of her teaching on students, rather than just enacting her teaching practices with fidelity. This brief example highlights how reflection on teaching practices, *retrospectively*, through biweekly meetings allowed Beth to modify her teaching practices, because it changed what she attended to or noticed *prospectively* while teaching.

Lifelong Learning

Self-regulation refers to how individuals plan, monitor, and reflect on their learning practices (Zimmerman, 2002). While self-regulation can occur over short timescales, often it is used to refer to the development of lifelong learning skills, which are developed over months or years. This relates to a number of practices, such as: goal setting, time management, study habits, collaboration, use of resources, and response to setbacks or failure. The way that individuals engage with these global skills has a profound impact on their learning, persistence, and success. Like at the more micro levels, by reflecting on their learning experiences *retrospectively*, individuals can change the way that they engage in learning *prospectively*, ultimately supporting their success as learners.

To illustrate this process, another example from undergraduate calculus is provided (Gandhi et al., in press). In this context, introductory undergraduate students completed written reflections on their learning experiences each week. At the end of the course, students also completed a final reflection. These reflections focused on lifelong learning skills, related to issues such as time management, persistence, use of resources, and collaboration. The goal was that these *retrospective* reflections would help students reflect *prospectively* about how they were approaching learning moving forward in the course. The students also worked with a framework of “grades as measurements,” which was intended to help them interpret their learning experiences. To illustrate this process, the example of one of Micah’s retrospective reflections is given.

After receiving low grades on a midterm, Micah chose to drop introductory calculus. As a student with a disability, Micah attributed the low grade at least in part to inadequate use of appropriate accommodations, rather than as a true indicator that they were not capable of succeeding in calculus. This perspective ultimately supported prospective reflection, as the second time Micah took calculus they were able to approach and interpret the experience differently. Micah noted (Gandhi et al., in press):

> The test is a tool that I use to measure my knowledge but for me, being a disabled student, is also a source of error. There are a lot of uncertainties: the writing of the test, the amount of accommodations that I have…and the availability of my textbooks being in an accessible format. These systematic biases lead to my tests being a composite score of how I am advocating for myself in terms of accommodations, my knowledge of the material, and the educational institution's ability to accommodate me and other disabled students.

Here Micah's reflection provides an interpretation for the grades they received; they are not a true reflection of knowledge, but rather a composite of knowledge and the biases that result from Micah having a disability, what accommodations are available, and how they are accessed. Micah was able to use this retrospective reflection as a means of interpreting specific experiences, which then guided future interpretations through prospective reflection. Here Micah describes the decision to drop Calculus I:

> I understood how to do calculus but I just didn't have the necessary tools to "do the experiment," [so to] speak…Finally, after being recommended multiple times by the DSP [Disabled Students' Program] staff to drop the class, I did. It felt like I was giving up. The experiment had failed…When November came around, I knew I needed to get my accommodations ready for [the Spring] semester. I emailed the DSP staff and the California Department of Rehabilitation and coordinated the accommodations that I needed for the next semester. I would take [Calculus I] again and attempt to succeed. I would rerun my experiment correcting for error. During the remainder of the Fall semester, I focused on getting ahead for [Calculus I] and coordinating my accommodations.

Micah was not successful the first time that they took Calculus 1. Initially, they took this as a sign that they were not capable of succeeding. However, using the combination of retrospective reflections *and* the framework of grades as measurements, Micah was able to reinterpret this lack of success; it was not related to lack of ability, but rather lack of appropriate use of accommodations. This shift in perspective changed how Micah approached Calculus 1 the second time around, prospectively. Rather than trying to "study hard" or spend additional time on the course, Micah instead put their effort on accessing appropriate accommodations. By helping Micah reinterpret the situation, in retrospect, reflection supported Micah to take a new approach, in prospect.

Discussion and Conclusions

Reflection is a key part of learning mathematics. Yet, work in mathematics education has focused almost exclusively on the development of concepts, rather than practices. This theoretical paper broadens the scope of research on reflection to include a focus on the development of practices in addition to conceptual understanding. To do so, it introduces the concepts of reflection cycles, prospective reflection, and retrospective reflection. *Prospective* reflection guides an individual's actions during the cycle, and the learning throughout the cycle is consolidated through *retrospective* reflection. Through appropriate engagement in retrospective reflection, an individual's prospective reflection in subsequent cycles is modified. As such, explicit efforts to promote retrospective reflection can be designed to support students to reflect prospectively in the future. This allows students to be more effective and strategic in their engagement as learners. In other words, having students regularly reflect *retrospectively* helps them develop lenses that they later use to guide their work *prospectively*.

Reflection cycles occur at various timescales, ranging from micro to macro. In each case, an individual's prospective reflection guides what they attend to and how they act in a given situation. At the end of an experience, explicit retrospective reflection is a mechanism for influencing the prospective reflection that takes place in a future reflection cycle. As such, reflection cycles highlight the important relationship between these two types of reflection for modifying one's mathematical practices. Moreover, at a theoretical level, the concept of reflection cycles provides a framework for seeing the unity in a wide variety of distinct but related reflective practices. This paper focuses on metacognition, self-assessment, noticing, and lifelong learning, but the framework may be applied to other practices as well. Rather than seeing these four different practices as entirely distinct, they can all be understood through the lens of reflection.

This paper contributes to reflection in mathematics education in a number of ways. First, the emphasis on reflection for concepts *and* practices broadens the scope of discussion about reflection in learning mathematics. Second, it provides a new analytic lens for making sense of existing studies in mathematics education. The focus on both *retrospective* and *prospective* reflection highlights that all reflection is not the same, and different types of reflection activities can be designed to support one another. Finally, this framework unifies a number of reflective practices, and can likely be applied to others that were not discussed above. As researchers begin to apply these concepts *prospectively* in the design of new research studies, it should open up new lines of inquiry related to reflection in mathematics learning.

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