BECOMING A MATHEMATICAL AUTHORITY: THE SOLUTION LIES IN THE SOLUTION

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This paper focuses on the development of three skills underlying mathematical authority: (1) explanation, (2) justification, and (3) assessment. An intervention was designed to help students develop these skills through explicit engagement with assessment in the classroom. Preliminary results from this ongoing study indicate that students had improved meta-level understandings of solutions, which supported greater levels of explanation in their solutions of problems.

Keywords: Design Experiments; Instructional Activities and Practices; Metacognition

Introduction

When a mathematician solves a problem or submits a proof to a journal, he or she doesn't wonder whether or not the work is correct; he or she *knows* it is. Most mathematicians self-assess using highly-internalized mathematical standards. In contrast, mathematics students routinely submit assignments with little sense of how well they did, relying on their instructor to be the arbiter of mathematical truth. For these students, mathematical authority is something that exists externally to them. This paper is focused on how students internalize mathematical standards.

Mathematical authority relates to positioning and identity (cf. Boaler & Greeno, 2000; Engle & Conant, 2002), as well as specific skills and domain knowledge. This paper focuses on three mutually supportive skills, hypothesized to be crucial to mathematical authority (as conceptualized in Figure 1): (1) explanation, (2) justification, and (3) assessment. Mathematicians use these skills to derive authority from the logic and structure of mathematics (internalized authority), rather than relying on some other authoritative source like a teacher or textbook (external authority). These skills are widely recognized as part of the multi-faceted nature of mathematical proficiency (e.g., NCTM, 2000) with explanation and argumentation specifically emphasized by the Common Core State Standards (Common Core State Standards Initiative [CCSSI], 2010).

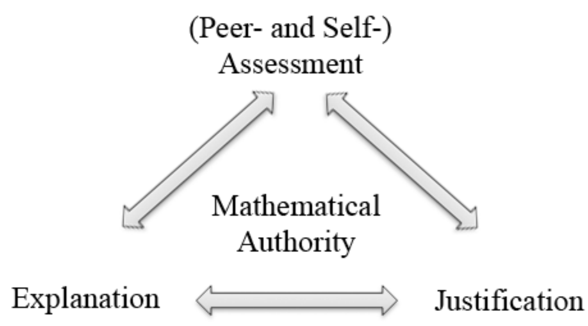
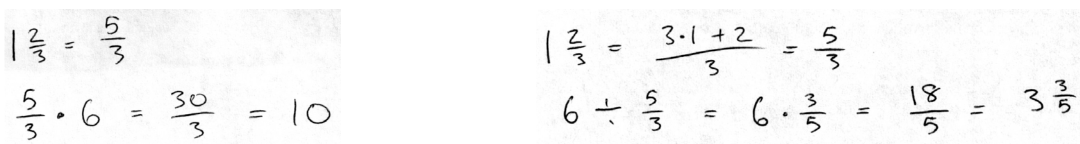


Figure 1: Three skills hypothesized to underlie mathematical authority

The design of this study's intervention draws on research showing that when students self-assess, they are unlikely to spontaneously generate information to test their understanding, which impedes accurate self-assessment (Dunlosky & Lipko, 2007). Just as perceiving constructive and deductive geometry as unrelated hinders mathematical performance (Schoenfeld, 1988), I hypothesized that perceiving explanation and justification as extraneous parts of a solution inhibits accurate self-assessment. When students are unable to use their own reasoning to justify their work, they are forced to rely on an external mathematical authority. Sadler (1989) suggests that standards of a high-quality solution should be communicated through a combination of descriptive statements, exposure to exemplars, and direct evaluative experience. Crucially, as students analyze others' work, they develop the required objectivity and skills to assess their own work (Black, Harrison, & Lee, 2003). Thus, this study used peer-assessment to promote the development meta-level understandings of solutions that are crucial to self-assessment.

Methods

This paper draws on preliminary data collected during from an ongoing design research study with elementary algebra students (N = 20) at a community college in the San Francisco Bay Area. Data were collected from classroom videos, student written work, and the instructors' daily reflections. As a pre-test, students assessed sample written work of two hypothetical students solving the problem: "*If a tortoise is traveling at an average of 1 2/3 miles per hour, how long would it take the tortoise to travel 6 miles?*" (see Figure 2). Students were presented with two solutions sequentially, and after seeing each solution were asked to explain the hypothetical student's reasoning, and why it was correct or incorrect. Finally, students were asked to reconcile the two conflicting solutions, and explain how they could determine which solution was correct.



a. b.

Figure 2: (a) Initial sample solution; (b) Sample solution presented after initial assessment

As an intervention, students were introduced to a framework for assessing mathematical solutions. The framework emphasized that a solution should answer three questions for the reader: (1) What did you do?; (2) Why did you do it?; and (3) Did you do it correctly? These relate to three parts of a solution: (1) the execution, (2) the explanation, and (3) the justification. Guided by the instructor, students discussed features of high-quality solutions to generate a rubric based on the above framework. Students also engaged in various peer- and self-assessment tasks using the student-generated rubrics.

The results presented here document students' changes in their perceptions of solutions. Because the data are preliminary, and peer-assessment can be seen as a precursor to self-assessment (Black et al., 2003), this paper focuses on the development of understandings that would support self-assessment, but not their actual application to self-assessment. This brief report considers the development of a few focal students, to highlight trends within the larger data corpus.

Results and Analysis

In the pre-test, students articulated *what* steps the hypothetical students took to solve the problem, but could not explain *why* they took them, even when pressed by the instructor (e.g., "why did the student multiply rather than divide?"). When asked to determine which solution was correct, only one student generated an answer. This student recognized that if the tortoise was traveling faster than 1 mph, then 10 hours for a travel time was much too long, so therefore the first solution must be incorrect. Other students either responded that it was impossible for them to determine which solution was correct, or that they didn't know how to figure it out. Students asked the instructor to resolve the mathematics for them.

The pre-test provided the basis for classroom discussions about important qualities of a complete mathematics solution. In these discussions, students articulated that the sample solutions lacked detail, thus providing limited access to the hypothetical students' reasoning. Students were presented with a framework for high-quality solutions, and were guided to generate a rubric using this framework. Students suggested 8 important features of a solution, such as: a written statement explaining why the solution path was chosen, checking units, and estimating what a reasonable answer would be. Students then discussed how the sample solutions would have been easier to assess if they had these features. The instructor introduced 5 additional features of high-quality solutions to the class to complete the rubric.

Two weeks later, students were once again presented with one of the sample solutions from the pre-test (see Figure 1, sample a). After analyzing the solution using the rubric they had developed, students explained how generating a more complete solution would have helped their classmate. Some students, like Tanya, focused on specific solution features:

> *Tanya:* It would have helped him if he put the units down on his paper to check what to cancel out, since the problem gives you miles and miles per hour.

Tanya seems to understand that units are not just part of a complete solution, but actually a tool for problem solving, because they help determine which arithmetic operations are meaningful to perform. Other students, like Enrique and Jason, focused on solutions holistically:

> *Enrique:* A more complete solution would have made him catch his mistakes.
> *Jason:* The execution is well done, but there's no explanation of any sort. The only thing that *seems* good is the answer.

Enrique's response emphasizes that careful solutions are important because they make our thinking (and thus mistakes) more evident. Jason alludes to the fact that the lack of explanation makes it difficult to say much about the student reasoning (e.g. "the only thing that *seems* good"). In sum, students transcended the specifics of the solution given, and exhibited meta-level understandings of solutions in general. These are the types of understandings that would allow students to begin to act as authorities themselves, rather than referring to an external authority.

As students develop a sense of high-quality mathematics solutions, it should also become evident in their written work. A comparison of students' solutions to the first two homework assignments (one week apart) provided evidence of such growth. (Note: the first homework assignment had 10 problems, and the second assignment had 11 problems but was of comparable length.) In general, solutions for the second homework assignment were more verbose and began to include explanations of reasoning (the first assignments contained little to no explanations). These changes were particularly striking for two of the students highlighted above, Jason and Tanya, whose solutions doubled in length (from 2 to 4 pages) between these two assignments. The increase in length was due to an inclusion of much more significant explanations and justifications in the second assignment.

Evidence of a more sophisticated understanding a solution was also evident in students' daily reflections. At the end of each class session, students were asked to answer a number of reflection questions, both in general and specifically related to the given lesson. When asked, "What does a good explanation in a math solution look like and why is it important?" Jason cogently responded:

> *Jason:* A good explanation can help someone understand the problem just by redoing the steps you took. After reading the steps they know why you took those steps and what you were doing.

This response seems to indicate a transition to seeing the solution to a math problem as an explanation of one's reasoning, not just "finding an answer." Jason's initial homework assignment included little to no justification or explanation, whereas his second homework assignment and responses to in-class questions were much more complete. Although we can only infer how Jason *sees* mathematics, there is evidence of changes in how he *does* mathematics. By explicitly turning students' focus to important features of solutions, it is possible to improve the quality of solutions that they submit.

Conclusion

By making the analysis of solutions an explicit focus of classroom activity, students were supported to develop meta-level understandings of solutions to mathematics problems. Students were able to articulate why including certain aspects of a solution can be essential, rather than an extraneous requirement imposed by the teacher. Evidence of growth was also apparent in students' homework solutions, which included greater explanations and justifications. Thus, preliminary results from this ongoing study show evidence of students' nascent development of internalized mathematical authority. These results provide the basis for the further refinement of classroom activities for promoting and studying students' development of skills of explanation, justification, and assessment. Moreover, the continuation of this work will allow for the study of students' application of these skills to self-assessment.

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