The Mathematical Imagery Trainer: From Embodied Interaction to Conceptual Learning

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ABSTRACT

We introduce an embodied-interaction instructional design, the Mathematical Imagery Trainer (MIT), for helping young students develop grounded understanding of proportional equivalence (e.g., 2/3 = 4/6). Taking advantage of the low-cost availability of hand-motion tracking provided by the Nintendo Wii remote, the MIT applies cognitive-science findings that mathematical concepts are grounded in mental simulation of dynamic imagery, which is acquired through perceiving, planning, and performing actions with the body. We describe our rationale for and implementation of the MIT through a design-based research approach and report on clinical interviews with twenty-two 4th–6th grade students who engaged in problem-solving tasks with the MIT.

Author Keywords

Educational technology, mathematics education, embodied cognition, Wii remote, design-based research.

ACM Classification Keywords

H.5.2. [Information Interfaces]: User Interfaces—input devices and strategies; user-centered design.

General Terms

Design, Human Factors, Theory.

INTRODUCTION

Human-computer interactions can be the site of rich, transformative cognitive development, including the construction of new concepts. As mathematics–education researchers, we study this generative space by analyzing students' learning progressions as they interact with computer-based, educational technologies.

Current instructional practices in mathematics education rarely leverage the learning opportunities presented by students' body-based experiences. Yet, research into the

Figure 1. **The Mathematical Imagery Trainer (MIT) in use by a 5th-grade student during a clinical interview.**

embodied nature of mathematical cognition implicates the critical role of concrete, body-based activity in grounding mathematical concepts and solving problems [2,21,26,29]. In particular, a pilot study conducted by Fuson and Abrahamson [16] demonstrated that students' *cognitive* difficulties in understanding proportional progression (e.g., the sequence of equivalent proportions 2:3, 4:6, 6:9, etc.) coincided with their *physical* difficulty in enacting such progressions with their hands (e.g., one hand rises vertically by 2 units per time beat while the other simultaneously rises by 3, beginning from a common baseline, such as a desk).

Oriented by the embodied cognition paradigm and the above research finding, we have designed, implemented, and evaluated a technology (see Figure 1) for providing students with an embodied experience from which to build understanding of the mathematical concept of proportional equivalence. Our design starts with the premise that basic mathematical concepts can be grounded [6] in dynamic imagery [36], a subclass of visual imagery (e.g., "pictures inside the head") [20], which is acquired from and encodes perceptual and motor experiences. For instance, "addition" sprouts from early experiences of stacking and grouping sets of objects together [21]. Moreover, mathematical reasoning, such as determining the angle at which a glass of

water spills or inferring rules for the rotation of gears, has been found to rely on the mental simulation (or "imagined action") of dynamic imagery [37].

Plausibly, certain mathematical concepts may be difficult to learn because our everyday experience fails to provide suitable dynamic imagery. That is, students who display difficulties in learning a mathematical concept may be otherwise ready to learn the concept, except they lack a suitable image for grounding the concept. Our broad aim is to help students acquire such missing imagery from contrived experiences that rely on computational media to "phenomenalize" [30] the abstract mathematical concept in the form of a concrete, physical challenge.

To evaluate the effectiveness of our design, and in order to deepen our understanding of the apparent roles of embodied interaction in mathematical learning, we have conducted a series of clinical interviews with twenty-two 4th–6th grade students. In these interviews, we monitored students' learning trajectories as they attempted tasks involving our instructional device. Starting with an *a*mathematical situation, we introduced a sequence of activities to shift the students' initial perceptions and reasoning toward mathematically normative practices and discourse [1]. Central to our approach is a pedagogical commitment to Papert's vision of constructionist learning in which *bricolage* antecedes hypothesis [28].

RELATED WORK

Learning and Dynamic Imagery
We chose the content domain of proportionality (e.g., 2:3 = 4:6) because the learning of rational number concepts has historically been fraught with conceptual impediments (for a recent review, see [14,22]). We focus specifically on proportional *progression,* because this content lends itself to challenging physical actions, as in the pilot study mentioned previously [16]. In that study, the authors asked a classroom of 5th graders to lay both hands on the desk and then raise one hand by 2 inches per beat while simultaneously raising the other hand by 3 inches per beat. Most students maintained a constant vertical distance between the levels of their rising hands, which would correspond to a progression such as 2:3, 3:4, 4:5, etc.

This physical demonstration of non-proportional movement foreshadows the most frequently demonstrated (and lamented) student error in the mathematics-education literature: the inappropriate use of additive reasoning in multiplicative situations. Students frequently and erroneously assume that "2/3 = 4/5" because of the additive/subtractive equivalence across numerators and denominators (i.e., "4 – 2 = 5 – 3") [5]. We conjecture that students' robust confusion stems from a lack of suitable dynamical imagery in which to ground, or against which to check, their nascent understanding of proportionality [29].

Unfortunately, conjectures pertaining to hypothetical cognitive constructs raise methodological challenges. A central difficulty in evaluating the roles of imagery in mathematical reasoning has been that these psychological constructs are currently inaccessible for direct measurement. Because we cannot see people's "pictures in the head," we instead rely on indirect means of investigating imagery. For instance, to examine how procedural fluency is deeply rooted in body-based experiences, Hatano et al. [18] asked abacus users to solve arithmetic problems in the absence of an actual abacus while also performing interfering finger-tapping tasks. Performance at "mental abacus" degraded for all but the most skilled abacus users, demonstrating how arithmetic reasoning relies on simulated imagery. This finding is corroborated by a related study in which Stigler [40] found that skilled abacus users self-reported visualizing an image of a real abacus while performing mental arithmetic.

Such experimental designs are well geared for studying imagery, because the researcher can confidently identify the experiential basis of the mental operations, and especially because the experience is based on interaction with a known object. However, cross-sectional studies, such as the above, do not examine individual learning *processes*, which are important in curricular design and in investigating the emergence of conceptual understanding.

Rather than seeking out an existing physical artifact, such as the abacus, we took a more proactive approach by directly inducing an unfamiliar image and then asking students to begin reasoning with it as its mathematical meanings emerged. Prior research has not examined the initial construction of imagery from embodied experiences nor how it begins to support mathematical reasoning, moreover in a single instructional session.

Our design is inspired by the notion of reflective activity, and, in particular, the notion of learning as a reflective conversation with materials. In their analysis of design and learning, Bamberger and Schön [6] posit that learning is necessarily *knowing-in-action*. Due to limited information-processing capacity, users of our design cannot, in advance of implementing a particular "move," consider all the consequences and qualities that may eventually be considered relevant to its evaluation. The immediate corollary is that some decisions emerge organically through the user's conversation with the material of a given system; that is, the user analyzes the system, plans and executes the move, then reflects on the (oftentimes unintended) consequences of this move—a process referred to as *seeing-moving-seeing*. The term "seeing" can be interpreted as sense-making; the user makes sense of the system by using it and reflecting on his actions. In our design, we have aimed to sequence the available user actions such that the consequences of one move may lead naturally to more pedagogically desirable moves in a trajectory from naïve to mathematical reasoning.

Embodied Interaction and Tangible Interfaces

Antle, Corness, and Droumeva [4] have used embodied-interaction designs to elicit, train, and apply users' embodied metaphors as a means of developing intuitive fluency with music creation. Their work extends the classical uses of metaphor in graphical user-interface design, such as the "desktop" metaphor, to whole-body, remote-action interfaces. Working with the specific metaphor of "Music is physical body movement," they implemented a computational system that helps children understand musical concepts such as melody, harmony, and rhythm in the form of intuitive, physical analogs (cf. [15]). Whereas our work shares many similarities in its design rationale, we are targeting a different underlying cognitive mechanism: dynamic imagery pertaining to the ubiquitous conceptual metaphor of "Quantity is spatial extension" [21].

Antle et al. [3] have also studied the role that embodied metaphor can play in supporting "reasoned imagination and learning." Their conclusion after analyzing several case studies is that theoretical expectations for embodied-interaction designs often do not match empirical findings in user studies. Whereas Antle et al. investigated open-ended and imaginative uses of their designs, our study employed a protocol to funnel users' interactions toward a specific outcome. While our participants occasionally explored the interface in unanticipated ways, the protocol of goal-oriented activities constrained interactions with our design into pedagogically desirable trajectories.

Cress et al. [13] used digital dance mats in a design for improving kindergarten children's fluency with relative numerical magnitudes. Their study built on the premise, supported in the literature, that basic numeracy reliably predicts future mathematical performance. In an experimental condition, children used whole-body gesture to solve a magnitude comparison task on a number line that was projected on the floor in front of the mat. A control condition presented a similar task on a tablet PC with input from an electronic pen. In a randomized, cross-over study of these conditions, children's improvement at the magnitude comparison tasks was greater for the experimental group that used full-body motion. This result supports our rationale for choosing embodied interaction as a promising means of promoting mathematical learning among children.

Although we find Cress et al.'s experimental design—a pre-to-post-intervention quantitative assessment of learning gains—to be appropriate for their research questions, our study seeks, rather, to understand the processes between the "pre" and "post." This includes nuanced interpersonal exchanges that occur in the middle of the activities themselves. We believe that qualitative analyses of *process* rather than only *product* constitute an important complement during the development of curricular designs [35].

Marshall, Cheng, and Luckin [25] conducted a study to measure adults' learning gains on a computer-based balance beam task while controlling for the effects of manipulation and agency. Their two-by-two experimental design varied both the type of interface (tangible vs. traditional, mouse-based) and the degree of learners' agency (self-directed vs. prescribed activities with the beam). They found no significant effect for either condition, and conclude that although this finding may not apply to other learning tasks and other groups (such as young children), it is a caution against assuming without rigorous proof that tangible interfaces are more effective at promoting learning than traditional interfaces.

A controlled experiment similar to Marshall et al.'s would be crucial for comparing our tangible interface against

a.   b.

Figure 2. **The MIT with IR-reflective tennis balls as hand trackers. A 5th-grade study participant "searches for green": having moved her left hand too high (see red screen in Figure 2a), she lowers it to effect a green screen (Figure 2b).**

Figure 3. **The MIT with hand-held IR emitters, which we found to be the most reliable means of hand tracking. A black matte surface on the desk helps reduce glare that can confuse the Wiimote's IR camera into seeing two IR sources. See http://tinyurl.com/edrl-mit2 for a 5 minute video clip showing the MIT in use.**

alternatives (even non-technological learning materials) to justify widespread deployment in schools. We believe that the successful design and deployment of educational technology at scale requires mixed-method assessments that include experimental designs. This approach is demonstrated by Roschelle et al.'s [32] deployment of SimCalc, a curricular design for entry-level calculus concepts that started as a design-based research effort similar to ours. We also note, however, that our instructional design encompasses more than just a tangible interface: the sequence of activities as mediated by the interviewer in our study is a critical component.

Finally, several HCI research studies have, like ours, used the low-cost Nintendo Wii remote as an input device for capturing complex hand movement and for implementing tangible or gestural interfaces. These include successful applications of hand-motion tracking for painting and drawing [23], music creation [8], and auto-racing and fencing simulation [39]. We extend those results to a new application area: mathematics education.

THE MATHEMATICAL IMAGERY TRAINER
Our instruction design leverages the high-resolution infrared camera available in the inexpensive Nintendo Wii remote to perform motion tracking of students' hands, similar to that described by Lee [24]. In our setup, an array of 84 infrared (940nm) LEDs aligned with the camera provides the light source, and 3M 3000X high-gain reflective tape attached to tennis balls can be effectively tracked at distances as great as 12 feet (see Figure 2). Lee reports that the camera has a 100 Hz refresh rate and a 45 degree field-of-view. In practice, we found the field of view to be slightly more restricted, requiring that we place the camera and LED assembly 10 feet from the student to

reliably capture a 3 foot window of arm movement. We oriented the camera on its side, since we required greater resolution (1,024 versus 768 pixels) along the vertical axis.

The Wii remote is a standard Bluetooth device, with several open-source libraries available to access it through Java or .NET.[1] Our accompanying software, called WiiKinemathics[2], is Java-based and presents students with a visual representation on a large display in the form of two crosshair symbols (*trackers*). The orientation of the 22" LED display (rotated 90 degrees and aligned to table height) and the responsiveness of the trackers is carefully calibrated so as to continuously position each tracker at a height that is near to the actual physical height of the students' hand above the desk. This feature is an attempt to enhance the embodied experience of the virtual, remote manipulation [10].

We found that in some cases detection of the reflective balls was too sensitive to the rotation caused by students' natural arm movement as they lifted the balls. This motion pivots the arms about the shoulder and consequently deflects the main reflector attached to the balls away from the axis of the camera and LED array. Our solution was to replace the LED array and reflective balls with battery-powered, hand-held IR emitters that the students point directly at the Wii camera (see Figure 3). With LEDs repurposed from generic TV remote controls, these emitters have a wide enough angle of operation to robustly capture students' hand motion.

DESIGN-BASED RESEARCH
The overarching approach of our study is that of design-based research [11,12], in which theory and design co-develop iteratively. Design-based research is not a methodology per se or a type of assessment. Rather, it is a disciplinary context within which we carry out empirical studies. As designers, we have the advantage of creating the instructional as well as the experimental design. We do so because we have a conjecture as to how learning could be better, yet current learning environments are unsuitable for addressing the conjecture; therefore, we design and evaluate a novel learning environment.

In our study, empirical data were gathered by conducting interviews using the semi-structured clinical technique championed by Jean Piaget, the founder of modern cognitive-developmental research. We spread the implementation of the interviews thinly (no more than two per day), such that from day to day we would be able to

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[1] Our software uses the WiiRemoteJ Java library, available from http://code.google.com/p/bochovj/wiki/WiiRemoteJ. A .NET library called WiimoteLib is available from http://wiimotelib.codeplex.com/.

[2] Source code is available through a BSD license at http://code.google.com/p/wiikinemathics/.

introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs.

These rapid-prototyping changes—all motivated by the goal of optimizing the pedagogical quality and empirical utility of our subsequent interviews—were based on fieldnotes, preliminary analyses, verbal transcriptions, minutes from our team's daily debriefings, and collaborative wiki postings. Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of participants. We gradually incorporated into the protocol any activities and prompts that arose during interviews and that, in debriefing, we evaluated as eliciting "researchable moments" from the participants. These were moments in which unexpected behavior from a participant suggested new theoretical constructs that we wished to test in subsequent interviews.

There is an inherent trade-off of control for exploration in design-based research. Because we modified our protocol and even our materials over the course of the study, we cannot compare among or draw conclusions for all participants. The scientific merit of case studies emerging from design-based research is often disputed on the grounds of generalizability: regardless of how rich, systematic, and extensive they may be, our studies present only a minor collection of data points sampled from a larger universe of potential cases.

Drawing on Robert Yin's [43] work, we offer an alternative view on this criticism. Yin distinguishes two types of generalization. Statistical generalization is the (rightfully) acknowledged standard approach where researchers sample individuals from a population (people, classrooms, cultural groups) in order to generalize to the larger universe. Yet when dealing with context-dependent activities, such as learning, in a particular community, we encounter a difficulty with statistical generalization. After all, the community under investigation is one of myriad human communities, and grounds for generalization to the larger universe of cultural groups tend to fall short with appeals to statistical generalization (cf. [34])

Acknowledging the utility of statistical generalization, Yin argues for a different kind of generalization, one that he terms "analytic." In analytical generalization, the focus is not generalization to the larger population from which a sample was drawn. The target is theory, and the endeavor is to refine theoretical constructs as their utility is explored and corroborated. In our work, we have found analytic generalizability well-suited to design-based research, which is itself characterized by its reciprocal iterations of design and theory.

With this approach to generalization, theoretical considerations inform the design, and the design, in turn, is used to further develop theory. From such a perspective, our research contributes to characterizing the larger population universe not through statistical inference but by refining our theoretical understanding. As Yin emphasizes, and particularly relevant to our work, analytic generalizabilty is a natural approach when phenomena and "context" have fuzzy boundaries, which is to say, whenever one studies learning in a rich setting.

CLINICAL INTERVIEWS

Participants

The empirical data presented and analyzed in this paper were collected at a private K-8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). In addition, we collaborated with the school principal, the head of general studies, and five mathematics teachers. Within each grade level (4th–6th), we grouped the pool of volunteering students according to three achievement levels as reported by their teachers (high/middle/low). Across these performance groups, we selected roughly equal numbers of students, balancing for gender, for a total of 22 students.

We preferred working with students whom the teachers had indicated as typically more disposed to communicate their thoughts articulately. These more verbose students were distributed almost uniformly across gender and achievement level. Whereas this bias in our selection of participants, as well as the by-and-large upper-middle-class demographics of the school, limits the generality of our conclusions, this initial stage of our research required dense real-time verbal feedback from the students as they interacted with the designed learning tools.

Protocol

Interviews took place in a quiet room within the school facility. Students participated either individually or paired with a classmate (duration: mean 70 min.; SD 20 min.).

The interviewer guided the participant through a sequence of activities by first explaining each activity and then monitoring the participant's performance and providing formative comments so as to ensure that the task was clear. The first activity was introduced with the simple instruction, "Make the screen green" (see Figure 4a—in the final protocol the screen initially bore no virtual elements at all). The background of the application is colored along a gradient from red to green depending on how close the hand heights are to the chosen a:b ratio. For instance, raising the hand trackers to 3" and 12" will turn the screen red or yellow for a 1:2 ratio (the color feedback is modulated by an adjustable tolerance control), whereas moving to 6" and 12" turns the screen green (see Figure 5). For the students interviewed in pairs, each student controlled one of the two hand-trackers.

Once the participants found an initial "green" position, the interviewer asked them to "find green somewhere else." If participants responded by "locking" the distance between their hands in a fixed interval and moving them up or down, the screen turned red. Only once they relaxed this
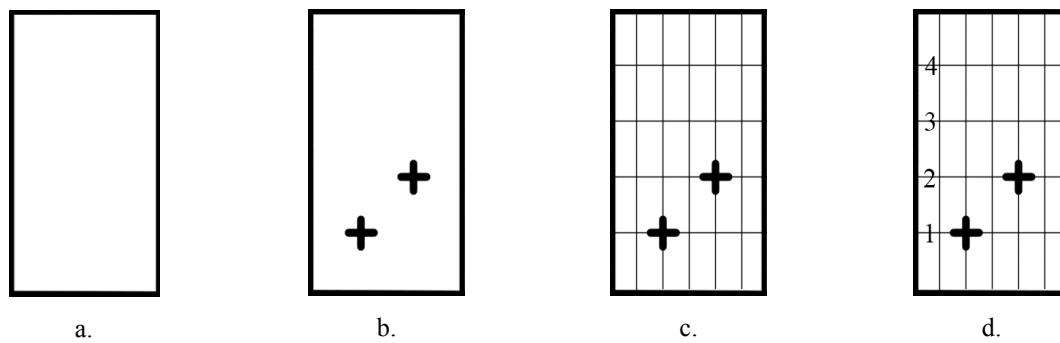
Figure 4. Four display configurations: (a) continuous-space mode; (b) continuous-space mode with crosshairs; (c) crosshairs with grid overlay; (d) crosshairs with grid overlay and *y*-axis numerals.

fixed distance between their hands and attempted to adjust it appropriately would they strike green again.

In the next activity, two crosshairs appeared on the screen to track the position of participants' hands (see Figure 4b). Participants might also at this point identify and articulate a rule to the effect that, *The higher you go on the screen, the greater the distance should be between your hands.*

Next, a grid appeared on the screen (see Figure 4c). The grid bears the capacity to shift participants' attention so as to re-construe each crosshair's location as its height above the base line—a height that can be quantified in terms of discrete units (e.g., 1 and 2 units, respectively). This may encourage a "snap to grid" strategy that utilizes the grid's inherent discrete-quantity relations, for example a recursive rule for transitioning from one green spot to the next: *For every 1 box you raise your left hand, raise your right hand by 2 boxes.* Note that this hand-to-hand relation is also a covariation, just as the height-to-distance relation, above, was a covariation. However, the values of the covariation have shifted from continuous–qualitative descriptors ("higher," "greater") to discrete–quantitative values ("one," "two"). Thus, though both covariations refer to the same hand motions—the objective stimuli—the meanings and planning of this physical enactment have evolved: the latter covariation is closer to normative mathematical practice and discourse for proportional equivalence, and in line with our pedagogical goals.

In the final mode, numerals appeared to the left of the grid (see Figure 4d, previous page), potentially alleviating a need for counting the grid boxes. Specific "green" numeral pairs, such as "3" and "6" in the case of 1:2, may evoke
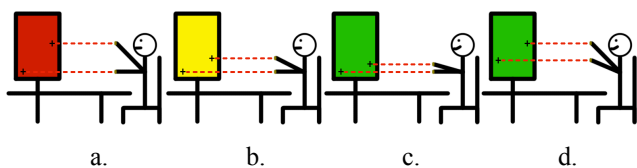

Figure 5. An example of continuous-space mode with crosshairs and a 1:2 ratio. A student exhibits (a) incorrect performance; (b) almost correct performance; (c) correct performance; (d) another instance of correct performance.

basic arithmetic operations and "facts," so that students recognize that the right hand should always be double (the height of) the left hand.

The interview ended with an informal conversation, in which the interviewer explained the objectives of the study, to help participants situate the activities within their school curriculum and everyday experiences. Finally, the interviewer answered any questions participants had, such as about our technology.

Results

All students succeeded in devising, performing, and articulating strategies for making the screen green. These spontaneous strategies bore logical and linguistic structures commonly observed in mathematical discourse pertaining to proportionality. For example, some students discovered, enacted, and stated the covariation of two constant rates (e.g., the left hand rises 1 unit per the right hand's 2 unit rise). We found variations in individual participants' initial interpretation of the task as well as in their trajectory through the protocol. However, by and large the students responded to the protocol items by progressing through similar problem-solving stages, with the more mathematically competent students generating more mathematical meanings and coordinating more among the quantitative properties, relations, and patterns they noticed.

Each student started either by working with only one hand at a time; waving both hands up and down in opposite directions; or lifting both hands up at the same pace, in continuous or abrupt gestures. Students soon realized that the actions of both hands are necessary to achieve green and that the distance between their hands was a critical factor. At first, though, most students anticipated they could achieve a continuously green screen by maintaining a *fixed* distance between their hands, as they moved both hands up and down along the screen, rather than by managing a *proportional* distance between the hands. For example, one 6th grade middle-achieving male student, Penuel, referred to this relation between the hands' respective positions as a "certain distance."

[15:32] Penuel: So it looks like... they have to be a certain distance away from each other for it to turn green.

[15:37] Interviewer: Uh-huh—

[15:37] Penuel: And if it's not a certain distance, it's not green—it's yellow or red.

Following further experimentation, Penuel proceeded to introduce another parameter into his reasoning about the distance between the hands, namely the hands' height above the desk.

[15:54] Int: Can you say anything more about the distance you're looking for?

[16:00] Penuel: I think it depends how far... how far they are from the ground.

Other students, such as Irit (5th grade; female; high achieving), realized that the fixed-interval rule was not optimal and articulated a changing-interval rule as a variant on the fixed-interval rule. (Note, below, the gentle prompt from the interviewer, who implicitly challenged Irit's assertion by drawing her attention to a phenomenal dimension she had taken to be constant.)

[19:26] Irit: Well, they are same distance, but like, away from each other. Like, uhm.

[19:35] Int: Uhm-hm?

[19:36] Irit: They're still the same distance away from each other, but they're just lower down here.

[19:45] Int: Huh... so what happens to the distance?

[19:52] Irit: Well… Oh! No, it gets shorter if you go down more, and then it gets tall…longer if you go up.

Along this same learning trajectory, another student (Liat; 6th grade; female; middle achieving) exhibited a telltale indication of conceptual development: a mismatch between her gestured actions and her verbal explanation [9,17]. Initially, Liat moved her hands in small fixed-interval gestures; the screen color would change away from green, and each time Liat would restore green with small jolting gestures. It appeared as though Liat implicitly assumed that her strategy enactment, rather than her strategy rationale, was at odds with the system (cf. [19]). Although the distance between Liat's hands was small lower down on the screen and dramatically larger farther up, she persisted in articulating her intended strategy (fixed interval) rather than her *de facto* enactment (changing interval).

[20:17] Liat: I think if I keep them apart and keep going up, it stays the same...

[20:23] Int: If you keep them apart and you keep going up it stays the same?

[20:31] Liat: It's not becoming red, but...

[20:36] Int: So... how are you thinking about keeping them apart? What is it that you are doing?

[20:41] Liat: Umm... I don't know.

Following the interviewers' prompts to "find green" farther up on the screen, Liat was finally able to generalize the rule that she had already objectively enacted.

[21:46] Liat: Oh maybe it's more… if it's farther up, then it has to be…they have to be more apart.

While most students initially located specific hand positions as stably producing green, several students also realized that there are infinitely many "green" hand-position pairs, so that in fact one can simultaneously raise both hands along respective continuous trajectories, a realization that some students articulated in terms of the hands moving at different speeds. For example, Penuel used the differing rates of his hand motions to deduce that a fixed-interval rule could not be correct and that, instead, the distance between the hands must increase with height.

[33:20] Penuel: I think the right hand has to move faster, while the left hand is kind of moving slower.

[33:41] Int: So what's going on?

[33:44] Penuel: Well, they obviously can't be at the same distance. But if I start here, and if the right one is moving, like a little faster, and it's going farther and farther away from the left hand, it will still stay green.

Another student, Siena (6th grade, female, low achieving), went on to formalize a rule for interaction in terms of both continuity and rate.

[15:13] Siena: Make the right hand go a little bit faster but... but let them both be at their own continuous pace.

The introduction of the Cartesian grid caused several students to shift their attention from the relative distance between their hands to the absolute height of their hands above the baseline. The pair Keith and Asa (6th grade, both male, high achieving) who were working with a 1:2 ratio, discovered and articulated a multiplicative relation for the heights in this way.

[30:41] Keith: Maybe it has to be double the number.

[30:44] Asa: Hmm, that's very possible. I'm going to be at one,... you're going to be at two. And then I'll be at two, and you'll be at four. And now I'm going to be at three...

[30:58] Keith: I'll be at six.

[31:38] Asa: You're totally right!

[31:39] Int: What is he right about? What is it?

[31:40] Asa: Uhm, the number has to be double.

Thus, they recast the rule "the higher, the bigger the difference" in terms of covariation between the absolute heights, that is, "for every 1 on the left, it's 2 on the right."

DESIGN ITERATIONS
Throughout our study, we iteratively modified the materials and protocol in light of new conjectures and theoretical constructs. Here, we explain the most important of these design iterations.

Passive versus active interaction

As an early prototype, we fabricated a mechanical device using rope and pulleys that physically moved participants' arms in a fixed 2:3 proportional progression, so that they could "feel" the progression. As opposed to the passive experience of hanging onto a rope, operating the MIT requires immediate agency, and so a question emerged as to how this agency should be framed, elicited, and guided. That is, what were participants to do with the MIT? What would be their task, how would they accomplish it, and what form of feedback might they receive on the quality of their performance?

Initially, we designed a task analogous to feeling the tension in the rope: participants used the hand trackers to closely follow two circular "targets" that moved in a proportional progression up the screen, with the background color indicating their accuracy. However, the first interview suggested that instead allowing "free-form" movement of the trackers was more theoretically compelling, because the ambiguity of the task better reflects what students must cope with in trying to make mathematical sense of the world around them. Figure 6 shows a schematic culminating from our brainstorming around this issue, which then formed the basis of the protocol for all subsequent interviews.

The numerical grid: scheme versus image

In later interviews, we added a new task at the point in the protocol soon after the grid and numerals were introduced (see Figure 4d). Namely, once students had expressed an *a*-for-*b* strategy—whether *b* was the increment in the right hand or the distance between the hands—the interviewer removed the grid and numerals from the display and asked students to resume searching for green.

Students' gesture and verbal utterances strongly suggest that they continued operating as though a grid were present. However, students' errors further suggested that their "mental grid" [6,13] was functioning more like a scheme [41] than as an image [20]. That is, the errors were not procedural but were expressed in imprecise magnitudes resulting in positional inaccuracies. As such, we may be witnessing the instrumental genesis of a utilization schema for proportionality [42]. In future work, we wish to better understand this phenomenon of the mind becoming equipped with mathematical instruments as the residual effect of working with objects [27], because we view it as paradigmatic of mathematical learning [33].

Overriding the hand trackers

Some students expressed a desire to force the trackers to move to precise height ratios that they inputted numerically. In response, we added a "driver" module to allow students to enter a list of heights for each tracker then animate the corresponding motion of the trackers (with corresponding background colors). In our seventh interview, a 4th grade student working with a 2:3 ratio began referring to the ratio as "one and one-and-a-half" and
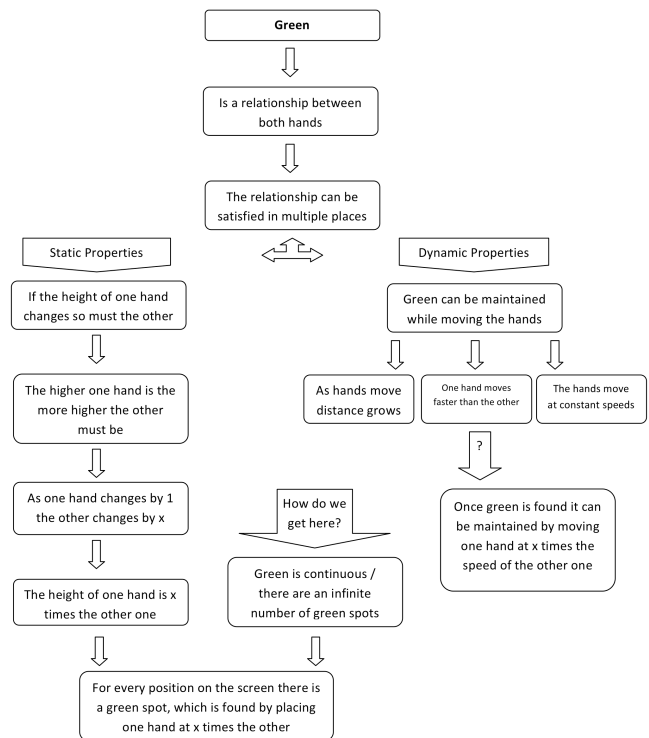

Figure 6. "What does green mean?" — a schematic created during the design process to conjecture on students' trajectories when given the task, "Make the screen green."

entered the sequence 1:1.5, 2:3, 3:4, 4:6 into the driver module.

The driver module was programmed to accept only integer values, and the input of a floating point value caused a runtime error! The student worked around the error by multiplying each ratio by 2 and entering the sequence 2:3, 4:6, 6:9, 8:12 instead. This left us with a designer's dilemma: should we fix the bug? Or had the bug created a "teachable moment" for exploring the decimal representation of rational numbers? Our solution was to fix the bug but add an additional switch that could "break" the input again by reverting it back to integer-only operation. This design decision mirrors the central principle of Schwartz's "Broken Calculator" [38], which has selectively disabled number or operation keys that force students to decompose and better understand arithmetic operations.

CONCLUSIONS

We have presented the rationale, design, and early results from the implementation and study of a novel educational technology. Remote manipulation, it appears, is more than just hand waving; it can be an opportunity for the mind to reflect on what the body can already do. Embodied interactions can drive both the realization and resolution of cognitive conflicts between users' implicit assumptions and their own observable enactment; and with careful guidance, these experiences can be recast in terms of emerging mathematical principles.

The interviewer played a critical role in the learning process, which we hope has not been understated. The MIT device could not stand in isolation, as an artifact separate from a knowledgeable instructor (for instance, in a museum installation). Our instructional design encompasses not only the MIT device but also an explicit protocol mediated by the interviewer. There will always need to be an educator providing similar support as the interviewer, and this is an important issue as we determine how to scale up our design to full-classroom use.

REFERENCES
1. Abrahamson, D. Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics 70*, 1 (2009), 27-47.

2. Alibali, M.W., Bassok, M., Olseth, K.L., et al. Illuminating mental representations through speech and gesture. *Psychological Science 10*, 4 (1999), 327-333.

3. Antle, A.N., Corness, G., Bakker, S., et al. Designing to support reasoned imagination through embodied metaphor, in *Proceeding of the 7th ACM Conference on Creativity and Cognition* (Berkeley CA, 2009), 275-284.

4. Antle, A.N., Corness, G., and Droumeva, M. What the body knows: Exploring the benefits of embodied metaphors in hybrid physical digital environments. *Interacting with Computers 21* (2009), 66-75.

5. Artigue, M. Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning 7*, 3 (2002), 245-274.

6. Bamberger, J. and Schön, D. Learning as reflective conversation with materials. In F. Steier (Ed.), *Research and Reflexivity*, Sage, London, 1991, 186-209.

7. Barsalou, L.W. Grounded cognition. *Annual Review of Psychology 59* (2008), 617-645.

8. Bott, J.N., Crowley, J.G., and LaViola, J.J. Exploring 3D gestural interfaces for music creation in video games. In *Proceedings of the 4th international Conference on Foundations of Digital Games* (Orlando FL, 2009), 18-25.

9. Church, R.B. and Goldin-Meadow, S. The mismatch between gesture and speech as an index of transitional knowledge. *Cognition 23* (1986), 43-71.

10. Clinton, K.A. Embodiment in digital worlds: What being a videogame player has to teach us about learning. Paper presented at the Annual Meeting of the American Educational Research Association (San Diego CA, 2004).

11. Collins, A. Towards a design science of education. In Scanlon, E. and O'Shea, T. eds. *New directions in educational technology*, Springer, Berlin, 1992, 15-22.

12. Confrey, J. The evolution of design studies as methodology. In Sawyer, R.K. ed. *The Cambridge handbook of the learning sciences*, Cambridge University Press, Cambridge MA, 2005, 135-151.

13. Cress, U., Fischer, U., Moeller, et al. The use of a digital dance mat for training kindergarten children in a magnitude comparison task. In *Proceedings of the 9th International Conference of the Learning Sciences* (2010), 105-112.

14. Davis, G.E. Fractions, ratio, and proportional reasoning [Special Issue]. *The Journal of Mathematical Behavior 22*, 2 (2003).

15. Farber, A. and Parker, L. Discovering Music through Dalcroze Eurhythmics. *Music Educators Journal 74*, 3 (1987) 43-45.

16. Fuson, K.C. and Abrahamson, D. Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase problem-solving models. In Campbell, J. ed. *Handbook of mathematical cognition*, Psychology Press, New York, 2005, 213-234.

17. Goldin-Meadow, S. and Singer, M.A. From children's hands to adults' ears: gesture's role in teaching and learning. *Developmental Psychology 39*, 3 (2003), 509-520.

18. Hatano, G., Miyake, Y., and Binks, M. Performance of expert abacus operators. *Cognition 5*, 1 (1977), 57-71.

19. Karmiloff-Smith, A. The child is a theoretician, not an inductivist. *Mind & Language 3*, 3 (1988), 183-195.

20. Kosslyn, S.M. 2005. Mental images and the brain. *Cognitive Neuropsychology 22*, 3/4 (2005), 333-347.

21. Lakoff, G. and Nuúez, R.E. *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books, New York, 2000.

22. Lamon, S.J. Rational numbers and proportional reasoning: Toward a theoretical framework for research. In Lester, F.K. ed. *Second handbook of research on mathematics teaching and learning*, Information Age Publishing, Charlotte NC, 2007, 629-667.

23. Lee, H., Kim, H., Gupta, G., and Mazalek, A. WiiArts: creating collaborative art experience with WiiRemote interaction. In *Proceedings of the 2nd International Conference on Tangible and Embedded Interaction* (Bonn, Germany, 2008), 33-36.

24. Lee, J.C. Hacking the Nintendo Wii Remote. *IEEE Pervasive Computing 7*, 3 (2008), 39-45.

25. Marshall, P., Cheng, P.C.-H., and Luckin, R. Tangibles in the balance: a discovery learning task with physical or graphical materials. In *Proceedings of the 4th ACM International Conference on Tangible, Embedded, and Embodied Interaction* (2010), 153-160.

26. Nemirovsky, R., Tierney, C. and Wright, T. Body motion and graphing. *Cognition and Instruction 16*, 2 (1998), 119-172.

27. Norman, D.A. Cognitive artifacts. In Carroll, J.M. ed. *Designing interaction: Psychology at the human-computer interface*. Cambridge University Press, New York, 1991, 17-38.

28. Papert, S. An exploration in the space of mathematics education. *International Journal of Computers for Mathematical Learning 1* (1996), 95-123.

29. Pirie, S.E.B. and Kieren, T. Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics 26* (1994), 165-190.

30. Pratt, D. and Noss, R. Designing for mathematical abstraction. *International Journal of Computers for Mathematical Learning 15*, 2 (2010), 81-97.

31. Radford, L. Gestures, speech and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning 5*, 1 (2003), 37-70.

32. Roschelle, J., Shechtman, N., Tatar, D., et al. Integration of Technology, Curriculum, and Professional Development for Advancing Middle School Mathematics: Three Large-Scale Studies. *American Educational Research Journal 47*, 4 (2010), 833-878.

33. Salomon, G., Perkins, D.N., and Globerson, T. Partners in cognition: Extending human intelligences with intelligent technologies. *Educational Researcher 20*, 3 (1991), 2-9.

34. Saxe, G.B. and Esmonde, I. Studying cognition in flux: a historical treatment of fu in the shifting structure of Oksapmin mathematics. *Mind, Culture, and Activity 12*, 3-4 (2005), 171-225.

35. Schoenfeld, A. H., Smith, J. P., and Arcavi, A. Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser ed. *Advances in instructional psychology*, Erlbaum, Hillsdale NJ, 1991, 55-175.

36. Schwartz, D.L. Physical Imagery: Kinematic versus Dynamic Models. *Cognitive Psychology 38* (1999), 433-464.

37. Schwartz, D.L. and Black, T. Inferences through imagined actions: Knowing by simulated doing. *Journal of Experimental Psychology 25*, 1 (1999), 116-136.

38. Schwartz, J.L. Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? *International Journal of Computers for Mathematical Learning 4*, 2 (1999), 99-119.

39. Shirai, A., Geslin, E., and Richir, S. WiiMedia: motion analysis methods and applications using a consumer video game controller. In *Proceedings of the 2007 ACM SIGGRAPH Symposium on Video Games* (San Diego CA, 2007), 133-140.

40. Stigler, J.W. "Mental abacus": The effect of abacus training on Chinese children's mental calculation. *Cognitive Psychology 16*, 2 (1984), 145-17.

41. Vergnaud, G. The theory of conceptual fields. In Nunes, T. ed. *Giving meaning to mathematical signs: psychological, pedagogical and cultural processes*. *Human Development* [Special Issue] *52* (2009), 83-94.

42. Vérillon, P. and Rabardel, P. Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education 10*, 1 (1995), 77-101.

43. Yin, R.K. *Case study research: design and methods*. Sage, London, 2009.