From tacit sensorimotor coupling to articulated mathematical reasoning in an embodied design for proportional reasoning[1]

Dor Abrahamson, José F. Gutiérrez, Rosa G. Lee, Daniel Reinholz, & Dragan Trninic
Embodied Design Research Laboratory, Graduate School of Education, UC Berkeley

Mathematics education designers and researchers are beginning to appreciate the pedagogical potential of embodied interaction (EI) instructional activities, yet little theory is available to understand its historical roots, sociocognitive mechanisms, and implementation practice. We draw on empirical data ($n = 22$) from a Grades 4-6 EI interview-based exploratory design for proportionality that involved "What's-my-rule?" remote-controlling of virtual objects on a computer display. Students were guided to reinvent mathematical concepts by reflectively instrumenting entrained perceptuomotor competence with interpolated symbolical artifacts. Analyzing qualitatively for critical dimensions, we propose and demonstrate three interaction milestones enabling or constraining learners' participation in conceptual EI activities: (a) re-seeing particular design features as peripheral-technological rather than central-mathematical; (b) re-seeing some would-be pragmatic actions as epistemic; and (c) re-seeing symbolic artifacts as affording or enhancing the enactment, explanation, or evaluation of hands-in immersive solution procedures. We delineate emergent principles for an effective EI mathematics design framework.

1. Introduction

1.1 What is Embodied Interaction?

Embodied interaction (EI) is a form of technology-supported training activity created, implemented, and researched by scholars interested in investigating multimodal learning. Through engaging in EI activities, users build schematic perceptuomotor structures consisting of mental connections between, on the one hand, physical actions they perform as they attempt to solve problems or respond to cues and, on the other hand, automated sensory feedback on these actions. One objective of EI design is for users to develop or enhance targeted schemes that undergird specialized forms of human practice, such as mathematical reasoning. As is true of all simulation-based training, EI is particularly powerful when everyday authentic opportunities to develop the targeted schemes are too infrequent, complex, expensive, or risky. Emblematic of EI activities, and what distinguishes EI from "hands on" educational activities in general, whether involving concrete or virtual objects, is that EI users' physical actions are intrinsic, and not just logistically instrumental, to obtaining information (cf. Marshall, Cheng, & Luckin, 2010). That is, the learner is to some degree physically immersed in the microworld, so that finger, limb, torso, or even whole-body movements are not only in the service of acting

---

upon objects but rather the motions themselves become part of the perceptuomotor structures learned. If to use David Kirsch's taxonomy, all EI gestures constitute epistemic actions, even if they are initiated as pragmatic actions (see Kirsch, 2006). In EI, the learner's body becomes concrete instructional material. EI is "hands *in*."

Typically, EI activities emphasize explorative perceptuomotor tasks and draw less on propositional or domain-specific reasoning (e.g., Antle, Corness, & Droumeva, 2009). Notwithstanding, EI activities may include standard symbolic elements, such as alphanumeric notation, diagrams, and graphs (Cress, Fischer, Moeller, Sauter, & Nuerk, 2010; Nemirovsky, Tierney, & Wright, 1998). Indeed, content-oriented EI activities are often designed explicitly to foster the guided emergence of domain-specific conceptual structures from domain-neutral perceptuomotor schemes.

1.2 Why Should Learning Scientists be Interested in Embodied Interaction?

This paper discusses documented evidence of students who struggle with core aspects of subject matter content through engaging in EI-based instructional activities. Such studies should be of particular interest to scholars who are developing models of learning informed by both cognitive and sociocultural theory (the "dialectical" approach, see diSessa, 2008), because the studies generate interaction data that bring out in relief mediation practices for the cultural signification of discipline-neutral schematic action. In particular, we are interested in theorizing how social interactions steer learners to leverage perceptuomotor competence in the appropriation of disciplinary forms of reasoning. Because our research is cast specifically within educational contexts of practice, including artifact design and actual teaching, we ask:

- What are effective educational practices enabling students to develop and then draw on perceptuomotor competence as a resource for conceptual learning? Specifically:

    o What are productive EI design principles? Given that computational environments enable designers to couple literally any physical action with any sensory feedback by interpolating algorithmic links between them (even if this coupling is ecologically atypical), which specific coupling would support the learning of some particular targeted concept?

    o What are the instructor's roles in orchestrating individual learners' development of perceptuomotor skill and then signifying it with mathematical meanings? In particular, what, if any, are the interaction challenges an instructor may encounter in supporting learners' articulation of their perceptuomotor skill? On the other hand, what, if any, are unexpected pedagogical affordances inherent to EI?

1.3 What Are Our Broader Research Objectives?

At its broadest, then, as learning scientists we are concerned with the general research problem of conceptual ontogenesis—how individuals develop higher-order ideas. The theoretical problem of conceptual ontogenesis is worthy of scholarly pursuit even just in terms of its critical implications for educational policy. In particular, this research consistently fuels and is fueled by heated policy debates over the controversial use of concrete objects and imagery in mathematics curriculum (e.g., Brown, McNeil, &

Glenberg, 2009; Schoenfeld, 2004; Sloutsky, Kaminski, & Heckler, 2005). In turn, this problem is informed by a spectrum of perspectives on cognitive and cultural sources of concept development (e.g., Carey & Gelman, 1991; cf. Newman, Griffin, & Cole, 1989).

We view our research paradigm as potentially illuminating of these theoretical and entailing policy debates over conceptual ontogenesis and the roles of media therein. Namely, prior research efforts stemmed from, and are positioned in distinct disciplinary traditions: they draw on diverse and even disparate epistemological assumptions with respect to the nature of teaching and learning and consequently use different, even incompatible methodology to pursue their intellectual or ideological agenda. As a result, we maintain, prior attempts to offer a comprehensive view of conceptual ontogenesis have by and large been encumbered or even precluded by partisan discourses that have rarely offered routes of reconciliation (see also Artigue, Cerulli, Haspekian, & Maracci, 2009, on the challenges of dialogue among diverse learning-sciences theoretical models of learning). Specifically, few studies on mathematical ontogenesis are "dialectical" (diSessa, 2008), in the sense that they attempt to model the nuanced and tight-knit reciprocity of cognitive agency in the social mediation process (but see, e.g., Sfard, 2002); few studies are multimodal, in the sense that they attend beyond verbal utterance also to action and gesture (but see, e.g., in Edwards, Radford, & Arzarello, 2009; Hall, 1996); so that fewer studies yet are both dialectical and multimodal in the sense that they offer models of how individual learners participating in cognitively demanding organized social activity build and sustain common-sense meanings for properties of objects they engage and scrutinize, even as they embody, inhabit, and signify mathematical practices grounded in these situated experiences (but see, e.g., Roth, 2009).

1.4 What Are the Methodological Affordances of EI for Research on Learning?

Arguably, EI creates powerful methodology for dialectical investigations of conceptual ontogenesis. To begin with, by augmenting the learning environment with computational algorithms and virtual objects, designers can engineer highly specified perceptuomotor coupling, such as coupling a precisely executed coordinated bimanual gesture with "correct"/"incorrect" color feedback, that are customized to support the instruction of hitherto challenging mathematical concepts in the spirit of reform-oriented pedagogical frameworks yet would be impossible or too awkward to implement using pre-electronic media. Thus, researchers enjoy opportunities to evaluate the merit of certain pedagogical frameworks with less implementation "noise." Moreover, recent technological advances in remote action now enable net perceptuomotor training, that is, "au naturel" interaction unencumbered by mediating instruments or interfaces whose acquired online operation taxes conceptually irrelevant cognitive resources. EI thus affords better opportunities to evaluate the potential of an approach to the design of mathematics learning materials that attempts to ground conceptual ontogenesis in onsite developed perceptuomotor competence. Finally, the very specific perceptuomotor schemes that EI activities foster are probably peculiar to those settings—we can quite safely trace their origin to our own design—thus lending greater validity to claims regarding the effect of particular interventions on learning outcomes. For example, any valuable mathematical insight that students manifest in the pedagogical context of stacking wood blocks could not be traced with confidence to that design alone, because children stack objects in regular extramural activities; yet these children rarely move their hands at different yet coordinated

velocities, so that a design supporting this new skill could be implicated as its probable source. The singularity of EI design may thus enable a comparatively uncontaminated tracking of ontogenesis.

1.5 What Are Our Specific Research Objectives?

Our objective in this paper is to offer some tentative dialectical reflections on conceptual ontogenesis—what it is grounded in and how it can be fostered. These reflections draw on a recent study, in which we investigated an instructional methodology for scaffolding the emergence of proportional reasoning from EI problem-solving activities of our design (Howison, Trninic, Reinholz, & Abrahamson, in press; Reinholz, Trninic, Howison, & Abrahamson, 2010). As reflective designers who believe that constructs, insights, and heuristics should guide, emerge from, and apply beyond the particularity of specific artifacts (diSessa & Cobb, 2004; Schön, 1983; Schunn, 2008), we hope with this work also to contribute more generally to the theory and practice of EI-based mathematics instruction, which we view as bearing great promise. Essentially what we are trying to accomplish in this exploratory study is to parameterize EI, which is still a fledgling practice with little educational exemplars or theory, toward building a comprehensive framework for the design, facilitation, and evaluation of pedagogical, concept-oriented EI activities. As such, we view as potentially useful to the field any coherent theoretical constructs and models we may offer that lend insight into how EI works, such as our characterization of pedagogical EI as first nurturing new perceptuomotor competence and then progressively cultivating it into conceptual structures.

The paper is structured as follows: after we lay out our claims, discuss EI's diverse origins, and explain the particular design we have been developing and researching, we present several empirical episodes from our early usability testing that we view as particularly illuminating of educational challenges and opportunities at the perceptuomotor–conceptual interface.

1.6 What Are the General Claims of This Paper?

Building on analyses of our empirical data, we argue that EI bears promise for the theory and practice of mathematics education. EI design, we contend, can create sites for presymbolic struggle with, and resolution of core conceptual challenges of pedagogically targeted subject matter content. Though these presymbolic perceptuomotor struggles are not "conceptual" in the classical sense of explicit association with particular disciplinary practice, including vocabulary, procedures, inscriptions, and broader institutional enframings, we view these struggles as critical to effective participation in these disciplinary activities. Our claim draws on empirically grounded theoretical views that the cognitive resources of mathematical reasoning are not symbolic propositions but presymbolic dynamical images (Barsalou, 2010; Goldin, 1987; Lakoff & Núñez, 2000; Pirie & Kieren, 1994; Presmeg, 2006), and we are further inspired by calls to ground mathematical learning in everyday intuition (Forman, 1988; Kamii & DeClark, 1985; Resnick, 1992; Thompson, 1994).

What form do these alleged presymbolic struggles take? We looked closely at our video footage of EI study participants as they were developing the design's targeted physical enactment. As we will elaborate in the empirical section of this paper, EI users develop new perceptuomotor competence by reinterpreting the *planning* and *executing* elements

of their interaction. In particular, users who initially enact a naïve theory-in-action for completing the task objective may interpret the automated "error" feedback they receive as indicating not an incorrect theory but rather execution imprecision. We use the terms "theory" or "theorem-in-action" to depict contextual problem-solving principles learners heuristically determine and subsequently entertain and evaluate as they investigate situations they attempt to control (Karmiloff-Smith, 1988; Karmiloff-Smith & Inhelder, 1978; Pufall, 1988; Vergnaud, 1983, 2009). Thus students' theory, which stems from their prior intuitions, is a double-edged sword (Cobb, 1989). That is, designers hope students will develop *new* theory in light of patterns in the automated feedback on their perceptuomotor enactment, yet students might be oblivious to these patterns and perseverate in confirmation-bias interaction. How then might students both draw on and qualify or modify their naïve theory?

We look at the roles of discursive and mathematical resources in guiding learners to re-see their own perceptuomotor enactment in pedagogically productive ways. Our analyses implicate the vital role of mathematical symbolic structures (e.g., a Cartesian grid) and inscriptions (e.g., numerals) available in a problem-solving space as the means by which students "re-describe" (Karmiloff-Smith, 1992) their entrained perceptuomotor enactment in appropriate disciplinary form. We demonstrate how these redescriptions can take surprising, pedagogically productive directions as students discover contextually advantageous ways of using the symbolic artifacts to enact, explain, or evaluate their task strategy. As such, we expand on neo-Vygotskian conceptualizations of appropriation (e.g., Bartolini Bussi & Mariotti, 2008; Saxe, 2004; Sfard, 2002) by implicating learners' creative *re*appropriation of symbolic artifacts in ways that are not mediated by the instructor yet are discovered in the discursive context as the artifacts' task-specific emergent semiotic–enactive affordances (cf. Hall, 2001; Neuman, 2001).

More generally, we submit, EI activities constitute rewarding empirical contexts for research aiming to deepen an understanding of how instructors discipline learners' perception of a shared domain of scrutiny (Goodwin, 1994; Hall, 1996; Stevens & Hall, 1998). Interestingly, the objects-to-think-with in our EI domain of scrutiny are not static and substantive but physical theory-in-action (Abrahamson & Howison, 2010).

1.7 Stepping Back to Move Forward: Recent Historical Roots of Embodied Interaction

The historical roots of EI in science and philosophy are numerous and varied. As a form of instructional activity, EI design emulates genetic epistemology (Piaget, 1968) and, in particular, its thesis that conceptual change is mediated by goal-oriented sensorimotor interaction followed by reflective abstraction on these interactions (for a converging view from "Constructionism," see also Papert, 1980, on "body syntonicity"). EI rationales can be further affiliated with broad principles of Phenomenology implicating immersed action as epistemically antecedent to reflective reasoning (e.g., Husserl, 1970). EI design is often inspired by grounded-cognition research and, notably, the empirically supported conjecture that human reasoning transpires not as processing *a*modal symbolically encoded propositions but as simulating fragments of previously experienced modal activity (Barsalou, 1999, 2010; Dourish, 2001; Hommel, Müsseler, Aschersleben, & Prinz, 2001; Varela, Thompson, & Rosch, 1991). That is, as some cognitive linguists have long been arguing, human reasoning—including mathematical reasoning—utilizes kinesthetic image schemas from everyday mundane activity that are projected as

conceptual metaphor to generate and communicate thought (Johnson, 1987; Lakoff & Núñez, 2000; Núñez, Edwards, & Matos, 1999). Curiously, the action-before-concept tenet from philosophy and cognitive science resonates well with sociocultural activity theories of knowledge mediation (Engeström, 1999; Wertsch, 1979) as well as with cultural–historical precedents of indigenous pedagogical practice (e.g., Kammer, 1986). Finally, because EI often uses remote action, a modality of particular interest to EI practices and research is manual gesture—hand movements that do not act upon actually present substantive or virtual objects but rather are associated with communication and reasoning, such as about acting upon non-present imaged objects. Laboratory studies repeatedly demonstrate the mediating role of gesture in the generation, teaching, and learning of disciplinary concepts and procedures; gesture has the capacity to evoke multimodal images, concretize emerging notions, implicitly communicate information and forms of reasoning, and frame perception (for a recent review, see Goldin-Meadow & Beilock, 2010; see also Yap, So, Melvin Yap, Tan, & Teoh, 2011). Gesture is thus relevant to analyzing discourse around EI, when participants mime their actions and tutors refer to these gestural performances.

In sum, the manifold roots of EI in scholarly exposition have converged and grown to a tipping point where, coupled with recent technological advances, EI stands to become a focus of design for, and research on mathematical learning. As design-based researchers of mathematical cognition and instruction, we consider EI activities useful empirical settings for research on the ontogenesis of mathematical concepts. These immersive activities create tension between unreflective orientation in a multimodal instrumented space and reflective mastery over the symbol-based redescription of this acquired competence (Nemirovsky, Tierney, & Wright, 1998). In particular, our research is centered on understanding how technologically instrumented procedural performance (Dreyfus & Dreyfus, 1999; Klemmer, Hartmann, & Takayama, 2006) gives rise to the enactment of sociocognitively normative mathematical practice ("mathematical signs" grounded in "artifacts signs," see Bartolini Bussi & Mariotti, 2008; "situated abstractions," see Noss, Healy, & Hoyles, 1997).

2. Design

Our design was propelled by a conjecture that EI activities might supplement everyday mundane activity technologically by creating opportunities for learners to develop image schemas that are vital to mathematical learning yet hard to come by outside of dedicated instructional contexts. Namely, we sought to build an interactive computational system supporting the entraining of proportional transformation as well as a set of mathematical instruments for students to rearticulate their emerging strategies in ways that would be useful for their understanding of, and fluency in proportional and related constructs.

Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"fixed difference" solutions (e.g., "2/3 = 4/5" - Lamon, 2007)—indicate students' lack of multimodal action images to ground proportion-related concepts (cf. Pirie & Kieren, 1994). We thus built a microworld wherein the physical solution operations inscribe the kinesthetic image schema of the emerging mathematical notions. Study participants were technologically steered to move their arms in a

choreographed form that would subsequently emerge through mediation as proportional. A sequence of mathematical instruments interpolated into the microworld supported the progressive mathematization of these actions (that is, their signification in formal mathematical inscriptions).

At the center of our instructional design is the Mathematical Imagery Trainer (hence "MIT" - see Figure 1 and Figure 2, below. For detailed descriptions of the device's rationale and technical properties as well as initial empirical results, see Reinholz et al., 2010; Trninic, Reinholz, Howison, & Abrahamson, 2010).
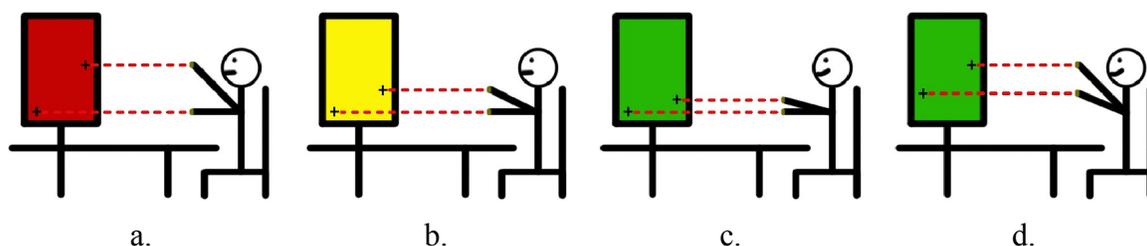

a. b. c. d.

Figure 1. MIT interaction schematics, with the device set at a 1:2 ratio, so that the right hand needs to be twice as high than the left hand: (a) incorrect performance (red feedback on exploratory gestures); (b) almost correct performance (yellow feedback); (c) correct performance (green feedback); and (d) another correct performance.

a. b.

Figure 2. MIT in action: (a) "incorrect" enactment turns the screen red; and (b) "correct" enactment turns the screen green. For a 5 minute video clip showing the MIT in action, see http://edrl.berkeley.edu/content/mathematics-imagery-trainer-video-pmena32.

The MIT measures the height of the users' hands above the desk. When these heights (e.g., 10'' & 20'') match the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. So if the user then raises her hands in front of this presymbolic "What's-my-rule?" artifact in accord with the ratio (e.g., to 15'' & 30''), the screen will remain green but will otherwise turn red (e.g., to 15'' & 25'', a coordinated hand elevation that maintains fixed distance of 10''). Study participants were tasked first to find green then to maintain it while moving their hands. The protocol included layering a set of mathematical artifacts onto the display, such as an adaptable Cartesian grid (see

Figure 3, below), to catalyze progressive mathematization of emergent strategies.

Participants included 22 students from a private K–8 suburban school in the greater Bay Area (33% on financial aid; 10% minority students). Care was taken roughly to balance students both by gender and by low-, middle-, and high-achieving groups as ranked by their teachers. Students participated either individually or paired in a semi-structured clinical interview (duration mean 70 min.; SD 20 min.). Interviews consisted primarily of working with the MIT. At first, the condition for green was set at a 1:2 ratio, and no feedback other than background color was given (see Figure 3a, below; we used this challenging initial condition only in the last six interviews). Then, crosshairs were introduced (see Figure 3b): these virtual objects mirrored the location participants' hands in the interaction space yet, so doing, became the objects users acted *on*, then *through*. Next, a grid was overlain on the display (see Figure 3c) to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions. In time, numerical labels "1, 2, 3,…" were overlain along the grid's *y*-axis (see Figure 3d): these enabled students to construct further meanings by more readily recruiting arithmetic knowledge and skills and distributing the problem-solving task.
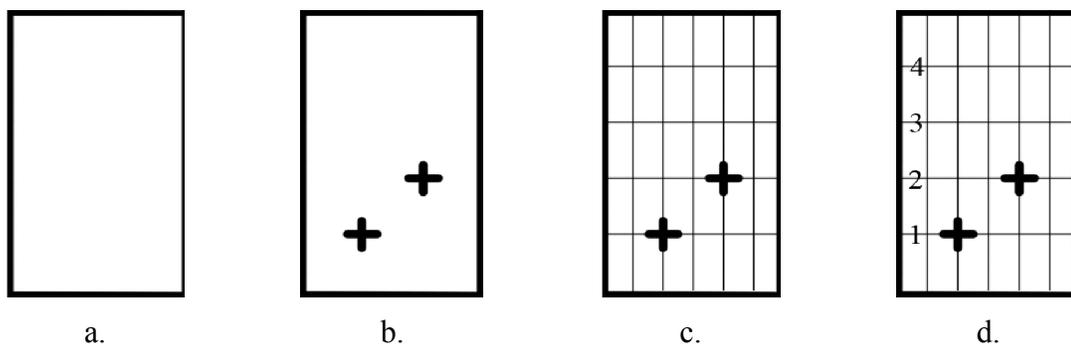

a.   b.   c.   d.

Figure 3. Display configuration sketches: (a) continuous-space mode; (b) continuous-space mode with crosshairs, i.e. the virtual objects that users manipulate; (c) crosshairs with grid overlay; (d) crosshairs with grid overlay and y-axis numerals.

Having briefly described our design and data-gathering methods, we now turn to discuss several features of conceptual EI activities that we discovered through collaborative microgenetic analysis of our data corpus (Schoenfeld, Smith, & Arcavi, 1991). In accord with elements of grounded theory (Glaser & Strauss, 1967), when we encountered in our video footage interaction behaviors of interest to our existing and emerging research questions, we sought to determine whether these behaviors were consistent within and between students in our larger data corpus. As a result, we winnowed out rare, though intriguing behaviors yet identified general cross-participant patterns. Nevertheless, our report on these patterns is located back in individual cases so as to lend authenticity to these patterns as they manifest in idiosyncratic contexts of inquiry and interaction.

3. Features of Conceptual EI Activities—an Empirical Account

This study has been our first attempt at designing for EI mathematical learning. The objective of this chapter is to describe, analyze, and generalize several features of our empirical data that we see as potentially paradigmatic of using these new media, with the hope that these narrative accounts, analyses, and constructs will prove helpful for other researchers working with these tools. Section 1 describes participants' struggles to reach entry-level engagement with our EI technology, Section 2 discusses mediated passages from tight perceptuomotor coupling with the technology to global proto-proportional schemes, and Section 3 relates students' creative reappropriation of mathematical artifacts in ways that extend their demonstrated use in pedagogically desirable directions.

3.1 From Peripheral to Mathematical Interaction

When instructional designers create artifacts, students may engage them in unexpected ways that lead them astray of the designers' intended interaction trajectory. This is true of innovative electronic artifacts in particular (Olive, 2000; White, 2008), possibly because these artifacts are historically new and so both designers and users are still developing an understanding of their affordances. For example, users of augmented-reality technology who manipulate tactile artifacts linked to virtual simulacra tacitly expect the virtual objects to interact in accord with their simulated physical ontology, yet those affordances may not have been engineered into the computational space (Hornecker & Dünser, 2009).

Users' unintended interactions with novel artifacts present nuanced issues for design-based researchers. On the one hand, a major concern of instructional designers is to gather empirical information on how prospective users interact with prototype products under cycles of iterative development. Moreover, design-based researchers are explicitly interested in users' idiosyncratic engagement with their artifacts, because the researchers view these diverse forms of engagement as hinting to the cognitive structures they wish to investigate and cater to (Abrahamson, 2009). Yet, on the other hand, design-based researchers are liable to focus their work on interactions that take place *in* their media of choice and make implicit assumptions that their users are able to meet prerequisite criteria of *entering* these media. By way of analogy, paperback writers assume that prospective consumers of their products can read and, a fortiori, that they are able to open a book and turn pages. Yet whereas designers distinguish clearly between para-media and in-media aspects of engaging their artifacts, and in fact often wish to make the para-media components "invisible" (Raymaekers, 2009, p. 2), this distinction may be entirely lost on users who are new to a medium, such as EI. New to a medium, users are unequipped to determine whether their failure to achieve task objectives is due to para-medium peripheral issues of operational fluency or in-medium substantive issues of inquiry heuristics, strategy, or content knowledge. Moreover, when artifacts under development are still "buggy" or "klunky," users can hardly be expected to navigate the phenomenological amalgam of designers' unintended and intended task features.

Arguably, a paper on mathematical learning need not trouble the reader with para-media interaction foibles and instead should highlight in the transcriptions only incidents of conceptually relevant interactions. And yet, given that EI activities are still quite new, we see value in sharing the user-experience (UX) consequences of our yet-imperfect technology. Thus, in this section, we compare two cases of interviewees who were either

successful (Asa and Kaylen) or not too successful (Boaz) in entering our EI medium, and we speculate on dimensions of interaction relevant to understanding this difference. In both cases, the tutors were graduate-student researchers, co-authors of this paper who were apprenticing as clinical interviewers using a semi-structured protocol, and the PI was present to guide and occasionally intervene.

We begin with Asa and Kaylen, 5th-grade male students indicated by their teachers as low achieving, and focus on Asa's reasoning at the introduction of the crosshairs. In this dyad-interaction task, the students each hold and manipulate one of the two tracker devices and attempt to co-produce a green screen. Prior to the layering of the crosshairs into their interaction space, the children were observed to alternate their perceptual attention back and forth between the hand-held devices and the computer display. They looked at their devices so as to note their relative positions (e.g., to maintain or change the distance between them), and they looked at the screen so as to note the color feedback (whether it were green, yellow, or red). Even when lifting their hands to the level of their eyes, such that the devices were in the same line of vision as the screen, the participants continued to adjust their gaze back and forth. The following transcript captures Asa's utterance prior to the introduction of the crosshairs.

> Asa: Oh I see. So I think what's going on is that they [referring to two hand-held devices] have to be the same far… [RH thumb and index gestures a vertical interval] the same distance away from each other [RH, still holding the interval, gestures toward Kaylen's device].

Asa thus offered what we refer to as a "fixed distance" manipulation conjecture, namely he believed that in order to move the handheld devices yet maintain a green screen, the interval between the hands should be invariant (see below Asa's "theory"). We also note Asa's use of the pronoun "it" in "keep it the same distance" that suggests Asa has objectified the vertical interval between the two devices as the thing to be controlled.

JFG reveals the crosshairs on the computer display

> JFG: Can you see that?
> A/K: Yeah.
> JFG: [briefly engages in technology recalibration] So let's see if these help us. [...] See if you can make the screen green.
> Asa: Well... So let's try to find green again. [A/K manipulate devices] There. Stop! Yeah. I think my theory's right. Oh wait, but we both have to be *moving*, 'cause when we stop...
> JFG: You both have to be moving? What do you…What do you mean by that?
> Asa: Seems so... Oh wait, no, that can't be possible. [to Kaylen] Stop! [screen remains green even though hands are stationary] Yeah it's not true. Go up.

Asa temporarily considered a new "green" theory—the hands must move continuously—but then immediately refuted this short-lived theory in light of empirical data.

> JFG: You think you can find another green [gestures to top half of the monitor] kind of up here…?
> Asa: My theory is obviously wrong.
> JFG: What was your theory before?

Asa: My theory was that there was a specific [RH thumb/index, as before, gestures a vertical interval] height that they have to be from each other but [waves hand] that's wrong [shakes head].

JFG: So what do you… What do you think now?

Asa: Uhhm, I think that the height slowly increases. [Kaylen concurs]

In contrast to alternating their gazes between the hands and screen prior to the introduction of the crosshairs, when the crosshairs first appeared on the screen the children instantaneously gazed at them and thereafter almost never looked at their handheld devices while operating them. This transition from *here* to *there* apparently enabled Asa to re-examine his previous fixed-distance conjecture; in turn, the greater precision afforded by this transition to the screen enabled Asa to experience empirically information that he then interpreted as rendering his conjecture "obviously wrong." Furthermore, the gestural–symbolic interfacing revealed one of Asa's implicit framings; namely, that the devices/crosshairs "both have to be moving" so as to effect a green screen. At the point in the interview when the crosshairs were introduced, Asa had interacted with the artifact for approximately ten minutes with only modest progress along the dyad's inquiry into the mystery device. Then, over a very brief period of time and without any expert modeling, Asa was able not only to disprove two of his conjectures (fixed difference, moving) but to produce a new conjecture, by which the distance ("height") between the crosshairs should increase as they move up. In sum, Asa rapidly and successfully forded/blended the gestural–symbolic divide and immediately availed of supplemental inquiry affordances in this virtually extended problem space.

Next, we discuss the case of Boaz, a 5th-grade male student indicated by his teachers as low achieving. The text below is a transcription from the early part of the interview, before the crosshairs had been revealed on the display. Boaz had been attempting to make the screen green, and DR asked Boaz to explain his green-making strategy. In his response, Boaz will be referring to a certain "camera." This is not the video camera recording the session, but a sensor mounted on a tripod, part of the remote-action technology that picks up the infrared signal in the LED beam emitted by the handheld device the child operates. As we shall see, Boaz's fixation on this amathematical device in his visual field impeded his incorporation of the virtual objects on the computer screen and thus delimited the scope and efficacy of his inquiry into the mystery device.

DR: Well, then, *how* did you make it green? [gestures toward computer displays]

Boaz: *Oh!* By... uhhm…

DR: Are you figuring the green?

Boaz: By, uhhhm... [swerves body to the left, orienting torso away from the screen and toward the sensor; points toward sensor with both palms] By… So, I guess these two things right here [points to devices] are showing the, uhmm [points to sensor] camera. Like [points back at devices] they're looking at the camera, and, I guess, those light things on it [refers to an obsolete LED array that is still mounted to the sensor, a technological relic of an earlier MIT design] each have a different color in each column. And so when they move it up or down or equal [mimes accordingly], each one, like, I'm, it's… it's pinpointing….

DR: So what would you do to make it green?

Boaz: I would... [operating in the lower range of the embodied interaction space, just above the desk, Boaz lifts and lowers the two devices independently, monitoring for color on the large screen; he stops when the screen turns green and holds his hands fixed at that position] try pinpointing it by moving them up and down 'til I made green.

DR: Ok. So…[DR takes a mimed cue from DA] What if you tried to lift your hands up like this [lifts both hands in *parallel* into the upper ranges of the interaction space] and keep it green? [Boaz imitates by lifting his hands up in parallel above his shoulder height] Do you know any way to do that?

Boaz: Oh! It would go out of the motion [gestures toward DR's laptop]. It would go out of the camera [gestures toward sensor].

In the course of explaining to Boaz the boundaries of the physical interaction space, DR inadvertently mimes *parallel* hand motions, which would not effect a green screen. DR intends for these actions to inscribe for Boaz the interaction space *in general*. Boaz, however, does not share DR's pragmatics intentions, and so he cannot differentiate between what DR takes to be *technical* and *mathematical* dimensions of the mimed action. Thus, when Boaz literally imitates the parallel motions that result in a red screen, he concludes that the approach must be *strategically* in error.

When the crosshairs were eventually revealed, albeit he began tracking them in his peripheral vision, Boaz was still physically and perceptually oriented toward the sensor. As his alternating "I" and "it" utterances suggest, his attention wavered between his hands and their projection, between a gesture and its inscription, and thus between the technical and mathematical aspects of the situation. In contrast to Asa, Boaz did not objectify the vertical interval between the two devices as the thing to be controlled prior to the introduction of the crosshairs—his attention was still fixated on his hands rather than their relational property. Consequently, the crosshairs appeared to Boaz as something supplementary to the central activity itself, apparently complicating the interaction; in contrast, for Asa they afforded a means of evaluating existing theories.

In sum, Boaz, unlike Asa, remained "stuck" between media, and his performance and understanding of their embedded mathematics suffered. Indeed, Boaz required direct assistance from the interviewers in "untangling" the para-medium and in-medium dimensions of the MIT. His difficulty in fording from gestural to symbolic media is not too unlike that of children in a Vygotsky's study, who "when asked to draw good weather….indicate the bottom of the page by making a horizontal motion of the hand, explaining, 'This is the earth,' and then, after a number of confused upward hatchwise motions, 'And this is good weather'" (Vygotsky, 1930/1978, p. 108). To these children, as to Boaz, the focus remains on the gesture rather than on its projected inscription.

Both the case of Boaz, in our study, and the child in Vygotsky's study illuminate developmental challenges in negotiating media divides. Namely, these cases collectively frame sociocultural processes of acculturating novices into representational practices as supporting learners' blending or hybridizing of embodied–gestural and virtual–inscriptional interaction orientations. Whereas this hybridity is crucial to cognitive–discursive empowerment and, thus, to valued participation in collective practices, the divide between the embodied and the symbolic may at times prove too great.

As designers, the heuristic image we sometimes invoke is of Janus, the mythological two-headed watcher of gateways and corridors. We find this image powerful, because it reminds us of the need to evaluate the artifacts we design and the learning corridors they provide from the perspectives both of an expert–designer and a novice–user. Through iterated design cycles, we strive to improve these gateways by creating technologies and protocols that afford invisible gestural–symbolical interaction blends.

3.2 From Local Perceptuomotor Coupling to Global Schemes

Naama is a Grade 5 female student indicated by her teachers as low achieving. In this section, we describe Naama's early attempts first to move her hands while keeping the screen green and then to articulate a stable rule for doing so. Ultimately, Naama was successful in determining the MIT's multiplicative rules, yet we interpret her behaviors along the way as illuminating the complexity of grounding mathematical concepts in EI activities. In particular, on the one hand we demonstrate how perceptuomotor fluency may form the basis for schematic reorganization, yet on the other hand we highlight possible difficulties in facilitating this process.

Once Naama had first succeeded in positioning the crosshairs so as to make the screen green, the interviewers encouraged her to try moving her hands while still keeping the screen green. In her initial attempts to do so, Naama would first move both hands at a fixed distance from each other, which resulted in a red screen, and then she would correct to green either by returning both hands back to the previous green position or by adjusting the distance between her hands. Naama was thus able to inch her way up and down along the display by progressing from each green position to the next one, alternating between fixed-distance simultaneous bimanual actions yielding red followed by mono-manual sequential adjustment actions yielding green. We interpret the fixed-distance gestures as expressing Naama's theory-in-action. Because the theory was robust, Naama interpreted the "error" feedback (red screen) as indicating not a problem with her theory but production imprecision that does not bear on the theory-in-action but only on her perceptuomotor acuity and dexterity. That is, Naama tended implicitly to cast her adjustment manipulations as pragmatic "noise" irrelevant to the epistemic "signal" informing the evaluation of her theory.

Soon after, however, Naama appears to have noticed new properties of the situation—she accordingly reverted to a new strategy, which she explained thus:

> The higher you go [right hand/crosshair], it [left hand/crosshair] has to follow, kind of…. If you want it [hands/crosshairs] to go higher, this one goes higher [right hand jolts twice upward] and then you have to move this one a little bit each time you move [left hand jolt once upward].

Interestingly, whereas Naama does not verbalize the *quantitative* properties inherent to the "two" vs. "one" upward jolts she gestured, these two-vs.-one jolts anticipated the 2:1 ratio setting of the device. Indeed, this right-goes-up-two-units-then-left-goes-up-one-unit sequential strategy proved quite successful in terms of maintaining a green screen, barring the brief red interims. Naama was thus shifting from incorrect to correct theory-in-action—from: (a) simultaneous bimanual action maintaining fixed, unverbalized distance between the hands; to (b) sequential right-then-left quantified hand motions. This strategic shift indicates progress along a learning trajectory toward articulating an *a-*

per-*b* proto-ratio conceptual structure (e.g., "two per one").

Then, asked again to explain her strategy, Naama reverted to the naïve theory-in-action:

> You have to keep moving the hands and keep them in the same position.…kind of hold them in the same place.…you don't move your hands out of the position.

Executing this naïve strategy, however, again required the performance of adjustment actions. But here we witness dissociation between the mathematical and performance value of strategies, a dissociation that we view as offering challenges for EI design: Whereas Naama's theory was mathematically naïve, and we were keen for her to switch to the other strategy that *we* appreciated as mathematically more advanced, Naama was eager to pursue the naïve strategy, because she was becoming increasingly dexterous at performing perceptuomotor cycles, each consisting of a tiny fixed-distance jolt followed by a tiny adjustment. Naama appeared to prefer the naïve strategy—it was more practical.

One of the interviewers then asked Naama to lay down the tracking devises upon the desk and demonstrate her strategy using her bare hands. Note that by gesturing *sans* devices, Naama cannot receive online automated feedback on her gestures. One might therefore expect that Naama would move her hands at a fixed distance in accord with her explicit strategy and not perform any secondary adjustment actions, because there would be nothing to adjust by or to. However, and to our great surprise, whereas Naama announced she were moving her hands at a fixed distance, she simultaneously raised her hands miming a changing-distance action that approximated a correct 2:1 growth! We asked her to repeat the demonstration, and she did so twice, still insisting that her hands were at a fixed distance. What are we to make of this acute gesture–speech mismatch? It could indicate that Naama knew more than she could as yet say; that she was prepared for conceptual change (see Alibali, Flevares, & Goldin-Meadow, 1997; Church & Goldin-Meadow, 1986). That is, it appears that the EI activity entrained Naama to embody a pedagogically targeted action pattern before she could articulate it.

The desired outcome of EI conceptual learning activities is of course not a physical performance per se, but the development of an articulated scheme governing relations between key properties of this performance. In particular, our design was for participants to verbalize initially the relation between the hands' height and distance ("the higher you go, the bigger the distance") and then between the two hands' respective increments ("for every two we go up on this side, we go up one on this side"). In accord with this design objective, once the grid and numerals had been introduced onto the display, Naama orally likened the upward vertical gestures to "stacking blocks," two blocks on the right for every one block on the left. Whereas this verbal assertion echoed her earlier two-per-one gestured utterance, it arrived only after enduring manifestations of the fixed-distance strategy. Ironically, introducing the grid and numerals into the problem space— intervention actions that ostensibly should have empowered Naama—may have impeded her reflective abstraction on the budding "higher–bigger" embodied insight. Moreover, this premature instrumentalization of Naama's working space may have strengthened her fixed-distance strategy, because these mathematical instruments provided her with a familiar context to verbalize her naïve strategy arithmetically. In sum, whereas EI-induced perceptuomotor competence is necessary, it is not sufficient: embodied theory-in-action that remains uncultivated may never contribute to mathematical understanding,

because learners may revert to safe, workable "detour" solutions that do not navigate unfamiliar territory and are thus less cognitively challenging.

Currently, we can only conjecture a microgenetic account of how learners operating the MIT shift from fixed-distance to changing-distance theory-in-action. Namely, we do not know how, if at all, they reinterpret their adjustment actions as challenging their fixed-difference theory-in-action. We can only speculate that the learners become cognizant of consistency in the direction of the corrective adjustments: as their hands ascend, they notice that effective adjustments are always those that make the distance larger, not smaller. An alternative yet not mutually exclusive speculation is as follows. As learners become more proficient in engaging the tight local perceptuomotor loops and, thus, perform the overall gestures *faster* as well as draw less on *cognitive resources*, the overall figural pattern becomes more salient to them, and this global pattern then obtains to the local loops, which in turn become more efficient, and so on recursively, until the changing-distance action becomes too compelling to misinterpret. And yet, as we have witnessed, a robust naïve theory can go a long way in blinding learners to the "truth"…

More broadly, the discrepancy between Naama's embodied and articulated knowledge—between what Naama "did" and "said"—bears on our understanding of the technologically entrained perceptuomotor loops (action–feedback–adjustment) and in particular how these differ from a pedagogically targeted scheme yet contribute to its construction. For the child to endorse adjustment actions as epistemically inherent to a new emerging scheme rather than as pragmatically corrective of execution incompetence, the child has to disengage from the sequence of micro-perceptuomotor loops and engage in reflective abstraction on this activity (Karmiloff-Smith, 1992; Piaget & Inhelder, 1969)—the child has to switch from the local "trees" and see the global "forest." Yet, paradoxically, in doing so the child has recourse only to existing schemes as resources for building a new theory (Karmiloff-Smith & Inhelder, 1978). The child will therefore hold on to this theory until empirical counter-evidence becomes so salient it compels reevaluating the theory (Karmiloff-Smith, 1988). The new theory is initially articulated as a variant on, or in contradistinction to the existing theory, so that multiplicative reasoning could emerge as qualifying additive reasoning. It is thus that a collection of infinitesimal micro-perceptuomotor loops may become "ironed out" and integrated globally in schematic redescription vis-à-vis an emerging empirical correlation.[2]

Finally, as instructional designers we are stuck by phenomenological similarity of direct control of concrete instructional artifacts ("manipulatives") and remote control of virtual objects; by these actions' similar exploratory and discursive pedagogical affordances. Moreover, bimanual gesture itself might be viewed as an object-to-think-with—an immaterial action–object for sure, yet a thing that nevertheless can be reproduced, refined, referred to, analyzed, quantified, elaborated, debated, negotiated, and signified. That said, Naama's case is testimony to cognitive and interactional challenges inherent to

---

[2] Inquiring into the epistemological paradox of perceiving unfamiliar structure in the absence of explicit guidance, C. S. Peirce wrote of "hypostatic abstraction" (Hoffmann, 2003). Our study participants had not only to perceive new structure but do so against the backdrop of a functioning structure (the naïve theory). Yet our participants did not work alone but rather with an instructor who tactically and pragmatically manipulated their actions so as to draw their attention to conceptually relevant properties in the perceptual field and thus render an alternative structure plausible.

helping students re-see gestured "objects" in light of new conceptual structure.

3.3 Hooks and Shifts: Reappropriating Symbolic Artifacts

The dual construct of "hook and shift" was developed during the current project in an attempt to characterize a surprising behavioral pattern of students who were observed to spontaneously bootstrap mathematical forms through appropriating artifacts in the process of problem-solving a challenging task (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, under review). A *hook* is a subjective and contextual affordance of a mathematical artifact that students recognize as they become cognizant of the artifact's availability in the course of solving a problem and communicating their solution. Specifically, students engage the artifact as an enactive, explanatory, or evaluative utility—as a way of better implementing their practical competence or as a semiotic means of objectifying their unarticulated strategy. A *shift* is the students' consequent unanticipated, unpremeditated, undemonstrated, yet more mathematically sophisticated view of the problem situation that emerges through using the artifact (see Schön, 1992, on "see...move...see"). We have identified a set of interaction parameters that may enable or hinder hooks and shifts in artifact-mediated discovery-based learning (Abrahamson et al., under review). The current section demonstrates an occurrence of hook-and-shift. Albeit, at this time is it not clear how one might view this pattern as *unique* to EI design. Whereas the primary objective of this section is to enrich and ground our definitions of hook-and-shift, the *dyadic* interaction of the specific data episode selected for this section offers a supplementary, if complicating, dimension. Yet we believe this added complexity is ultimately rewarding, because its unique discursive qualities bring out in relief the very processes that we wish to demonstrate and understand.

Eden and Uri, two Grade 6 male participants, were selected for a paired interview on the basis of compatible mathematical achievement (both were identified by their teachers as high achievers). Their interview was conducted by an apprentice researcher (DT), with the lead researcher (DA) occasionally intervening. Eden and Uri were seated side by side in front of the MIT remote-action sensor system and computer display and each operated one of the two tracker devices (right-tracker device [RT] and left-tracker device [LT]). The students were presented with the task of making the screen green under an unknown 1:2 ratio setting (i.e., the screen would be green only when the right-crosshair [Rc] were double as high as the left-crosshair [Lc]).

3.3.1 Hooking to the grid. Prior to the introduction of the grid, Eden and Uri had been working together for nearly 11 minutes in the no-crosshairs (blank screen) condition and then another 7 minutes in the crosshairs condition. So doing, they identified two spatial dimensions—height and distance—as relevant to making the screen green and had articulated two theorems with regard to each of these dimensions: (a) Rc should be higher than Lc; and (b) the vertical distance between Rc and Lc is non-arbitrary. However, Eden and Uri disagreed as to whether this vertical distance should change or remain constant as the crosshairs move. Whereas both Uri and Eden observed different distances between the Rc and Lc at certain green locations, Uri interpreted this difference as a systemic principle for making green, while Eden attributed it to an HCI issue, as though the physical manipulation were inaccurate (Eden, apparently an avid video-game designer, referred to this error as the "human factor"). Uri articulated a covariant principle relating distance and height, explaining that "it has to get, like, farther away, the higher up we

are" and that "the lower you are, the less distance apart it has to be"—a *changing*-distance theorem-in-action. Eden, however, courteously responded with, "Well I'm not sure if it matters if you're lower or higher, but I think it's just, like, you stay the same distance apart"—a *fixed*-distance theorem-in-action.

Thus, Uri and Eden's collaborative hands-on problem solving enabled them each to notice and explicitly articulate a relation between the crosshairs' height and distance, yet whereas Uri concluded from their empirical data that the distance should vary, Eden concluded from the same data that it should not. This disagreement bore practical implications, because the dyad were co-operating the two devices—they each depended on the other to enact a green-making theorem-in-action, yet their respective theorems were mutually exclusive. Consequently, the students' success within this collaboration became contingent on whether or not they could rule between their incompatible fixed-distance and changing-distance theorems-in-action. At the same time, they apparently felt under-equipped to arbitrate in the continuous space. Namely, when the grid was subsequently introduced (see below), they recognized its potential for ruling between the theorems—they "hooked" to the grid largely for its discursive, argumentation, and arbitration affordances. Specifically, the grid served these boys to quantify the distance between the crosshairs and ultimately determine that this distance should in fact change between green spots, as Uri had believed. Eden soon concurred. The excerpt below begins immediately after DT had layered the grid onto the screen.[3]

> Eden: Grid.
> Uri: Yeah. [Grabs RT, lifts it, and remote-places Rc on the 1st-from-the-base-line gridline (hence "Rc up to 1-line"). Simultaneously, Eden, too, brings Lc up to 1-line. On the way up, between 0-line and 1-line, the screen flashes green for a moment but then turns red. Eden lowers Lc back down, holds it at .5 units. The screen turns green.] Oh so you can like show where… Let's see, so [Rc up from 1-line to 2-line]// if you're on here…
> Eden: //maybe it has to be two… [Lc up to 1-line (see Figure 4a, next page)] an entire box apart.
> Uri: [Rc up to 3-line] If I go here…
> Eden: [Lc up to 2-line; screen goes red (see Figure 4b, next page)] Then maybe you should raise it [Uri raises Rc to just below 4-line; screen flashes green]. So maybe the higher you go, the more boxes it is apart.
> Uri: Let's just say like I'm here [Rc down to 2-line], then he has to be one box under me…
> Eden: [Lc down to 1-line; screen goes green] And then the higher he goes//
> Uri: //and when I go here [Rc up to 3-line], he has to be like in the middle [Eden moves Lc up to 1.5 units; screen goes green]
> Eden: So the higher//
> Uri: //And here [Rc up to 4-line, while Eden moves Lc up to 2-line] he has to be like two boxes under me.
> Eden: So like the higher it goes, the more space there has to be between each. [Both Eden and Uri place their tracker devices on the table]

---

[3] RT = Right-Tracker device; LT = Left-Tracker device; Rc = Right crosshair; Lc = Left crosshair; // = overlap by next speaker.

Figure 4a. After the introduction of the grid, Uri (middle) and Eden (far right) find green with Rc at 2-line and Lc at 1-line, respectively. Noticing the distance between the Rc and Lc, Eden predicts that the fixed-distance subtends "an entire box."

Figure 4b. Immediately, Uri and Eden reposition Rc and Lc to 3-line and 2-line, respectively. The screen turns red. Upon noticing that the fixed-distance theory does not obtain, Eden says, "Then maybe you should raise it. So maybe the higher you go, the more boxes it is apart."

Thus, it appears that both Eden and Uri immediately appropriated the grid in view of its affordances to arbitrate among their conflicted theorems, however they differed with respect to the nature of their discovery, and this difference can be related to their idiosyncratic beliefs prior to the introduction of the grid. Namely, Uri had articulated a changing covariant relation between height and distance, so for him the grid afforded reiterating and quantifying this qualitative principle. Specifically, the grid enabled Uri to reformulate his continuous qualifier "get farther away" as the discrete quantifiers "one box" and then "two boxes." Eden, who had acknowledged the in-principle possibility of a changing-distance rule yet nevertheless maintained a fixed-distance rule, soon changed his mind and articulated a changing-distance hypothesis ("the higher you go, the more boxes it is apart"). However, Eden ends with a *qualitative* statement about "space," which suggests that he construed the grid as a means not of quantifying the "higher–bigger" conjecture but of evaluating whether or not this conjecture even obtained. Thus, Eden and Uri both hooked to the same artifact, yet they utilized it for different purposes. This episode demonstrates the plausible view that an artifact's subjective utility is contingent on the individual's goals. The episode also suggests that a dyad can engage in collaborative hands-on problem solving even as they hold different theorems-in-action (cf. Sebanz & Knoblich, 2009)

In the following excerpt, below, we will continue at a point where the dyad initiated further inquiry. As we shall see, the dyad's exploration will shift them from the now-consensual "higher–bigger" strategy to a proto-ratio *a*-per-*b* strategy. Both strategies can be viewed as expressing covariation—"the more *x*, the more *y*." However, the former strategy is continuous–qualitative, whereas the latter is discrete–quantitative, so that student adoption of the latter strategy is a pedagogically desirable outcome.

3.3.2 Shifting with the grid. Having reached consensus, Eden and Uri elaborate their explanation. They are now instrumented with the grid's quantification affordances, and this new conceptualization of space will engender the semi-spontaneous emergence of a new mathematical form. In particular, in the transcription that follows we will observe

that the students shift with the grid from the continuous–qualitative strategy to a discrete–quantitative strategy.

Uri: I think, like, uhm, when I go up to here [points to 2-line], he has to be one. Then when I go up//

Eden: Like for every… for every box he goes up, I have to move, go down[4]//

Uri: //You have to go up half//   //a box.

Eden: //Yeah//

The dyad's coordinated production of green tacitly modulated from simultaneous motions, in which the distance constantly increases and green coloration is maintained throughout, to sequential motions, in which each hand separately ratchets up to its respective designated destination and green is effected after a brief red interim, once the second ratcheted motion is completed. Imperceptibly, the linguistic constraints of the speech modality as well as the linearity of discursive turn-taking and ownership over actions thus shifted the dyad from their higher–bigger continuous–qualitative strategy to an *a*-per-*b* discrete–quantitative proto-ratio strategy.

Eden and Uri thus co-discovered that in order to maintain green, they should progress at intervals of ½ (Eden) and 1 (Uri) coordinated vertical intervals, either both going up the screen or both going down. It is through this serendipitous discovery that their earlier observation, "The higher you go, the more boxes it is apart," a covariation between height and distance, transformed (shifted) into a covariation that foregrounds the independent actions of the left and right crosshairs, "For every box he goes up—you have to go up half," a new strategy that is closer to normative forms for ratio (i.e., *a*-per-*b*). We wish to underscore that whereas the general *x*-per-*y* covariation form was maintained, its semantic–mathematical content was replaced (see Figure 5, below).

Mathematical Properties	"The more *x,...*"	"the more *y*."
Continuous–qualitative:	"The higher you go,…"	"the bigger the distance"
Discrete–qualitative:	"The higher you go,…"	"the more boxes it is apart"
Discrete–quantitative:	"For every box he goes up,…"	"you have to go up half"

Figure 5. "Covariation" linguistic structure across strategy shifts.

In addition to explicating our hook-and-shift construct, our analysis of the case has demonstrated that collaborative mathematical learning processes are impacted by nuances of personal/interpersonal framing to the extent of dissociation between a dyad's physical and epistemic actions. Namely, whereas the two dyad members collaborated on using a single symbolic artifact (the grid), their joint experiment simultaneously enacted an exploration of two different hypotheses ("same distance" and "different distance"). Uri was quite comfortable from the very onset with the higher–bigger principle, so he did not

---

[4] On several occasions throughout the interview, Eden used "up" to refer to the direction we would call "down" and vice versa. As it turned out, Eden is an expert video gamer. He thus may have been using schemes that are natural to video-gaming culture. Therein, and counter to non-gamer discourse, joystick actions are often mapped onto virtual actions such that swinging the joystick up-forward results in the first-person avatar looking or moving down.

need arbitration but refinement, whereas Eden, who had challenged Uri with a fixed-distance theorem, needed resolution. Once a common ground was established, the dyad was able to continue mathematizing the mystery artifact–phenomenon and jointly articulate a new mathematical form, which we recognized as pedagogically desirable.

4. Discussion and Conclusion

> The hybrid or the meeting of two media is a moment of truth and revelation from which new form is born….The moment of the meeting of media is a moment of freedom and release from the ordinary trance and numbness imposed….on our senses. (McLuhan, 1964, p. 63).

Even as learning scientists are increasingly accepting a view of mathematical reasoning as multimodal spatial–temporal activity, technological advances and free-market forces bode an impending ubiquity of personal devices capable of utilizing remote-embodied input. Poised between theory and industry, design-based researchers are only beginning to wrap their minds around the protean marriage of embodied cognition and remote action. Embodied interaction, a form of physically immersive instrumented activity, is geared to augment everyday perceptuomotor learning by fostering cognitive structures that leverage homo sapiens' evolved capacity to orient and navigate in a three-dimensional space, wherein the brain developed *by* and *for* action (Dourish, 2001).

We currently have far more questions than answers respecting the prospects of EI as well as principles for best design and facilitation of these innovations. Our strategy has been to engage in conjecture-driven cycles of building, testing, and reflecting. In this spirit, the current paper aimed to share our excitement with EI and offer some early observations and caveats. Yet, as often occurs when new media are encountered, we learn as much about *pan*-media practice as about the new medium per se. As such, the "ontological innovations" we have stumbled upon through our design-based research appear to bear more generally on how people do and could learn (cf. diSessa & Cobb, 2004).

4.1 EI Learning Affordances

Our analyses depicted EI learning as the evolution of users' subjective meaning for automated sensory stimuli and, in particular, the role of perceptuomotor skill as a vehicle or platform for the mediation of conceptual development. EI automated information (such as "green" in our design) evolved in the functional and cognitive roles it played. That is, green: (a) began as the task *objective*; (b) soon became the perceptuomotor *feedback*, as the users attempted to complete the task objective; and (c) came to hold together a set of otherwise unrelated hand-location pairs sharing a common effect and begging a name. Feedback on perceptuomotor performance thus came to demarcate the set of "green" number pairs as all belonging to an emerging *ontology*. In this sense, green functioned as more than an objective or feedback—it served as an ontological scaffold or conceptual placeholder. Ultimately, the scaffold collapses, or the placeholder is filled, once users determine the activity's mathematical rule and recognize the rule's power for anticipating, recording, and communicating the MIT's solution procedure. As one child gleefully quipped, upon determining the multiplicative relation of an unknown ratio, "I hacked the system!"

EI creates arenas for body-based mediation of qualitative mathematical notions. As the students engage in problem-solving our MIT mystery device, their physical actions inscribe a choreographed form with increasing deftness. Whereas the student views these forms as physical solution procedures, the educator—who views the forms from the vantage point of an expert's disciplinary perspective—conceptualizes these forms as multimodal image schema underlying the cognition of the targeted concept. Using representational resources and discursive guidance, the instructor may then steer students to progressively signify these image schema into what become conceptual metaphors of the emerging mathematical ideas. That is, even as the gestured forms lend meaning to mathematical propositions, they take on the epistemological role of metaphorical simulations (concept-specific "math kata," if you will). In practice, the mathematical concept emerges when students utilize new mathematical symbolic artifacts, which the instructor introduces into the problem-solving space, as means of enacting, enhancing, explaining, and/or evaluating their solution procedure. As such, mathematical knowledge emerges through recognizing a particular cultural form (e.g., the Cartesian grid) as contextually useful equipment (Radford, 2003). Specifically, students utilize these available forms to articulate their physical solution procedures, first multimodally (verbally and gesturally) and then also symbolically (by utilizing numerical inscriptions). Initially, these articulations are naive and qualitative, but they progressively adhere to mathematical forms through situated ascension from physical to mathematical.

In particular, we have been struck by learners' capacity to strategically re-appropriate available mathematical instruments in ways that have not been demonstrated for them yet bootstrap their conceptual practice. We have named this behavioral pattern "hook and shift," because the learners first adopt the artifacts to serve the enaction, enhancement, explanation, or evaluation of a current strategy yet through doing so they stumble upon the artifacts' embedded affordances that become revealed only through engaging them purposefully, so that the students reconfigure their strategy is ways that co-opt these powerful affordances. For example, note the case presented above where students, by engaging the grid, shifted from a "the higher, the bigger" strategy to the pedagogically advantageous proto-ratio and multiplicative strategies.

4.2 Toward a Heuristic Design Framework for Mathematical EI

Emerging from our research are the following empirically grounded principles of a heuristic design framework for EI mathematics problem-solving learning activities, that we can only sketch here: (1) The designer selects/engineers a learning environment that includes a device linking simple coordinated actions remotely to generic virtual objects; (2) The designer plans and implements progressive-mathematization supports in the form of layerable/removable symbolic artifacts; (3) Students' physical action should not only enable the gathering of data but actually constitute an integral component of the data; (4) Moreover, students' physical solution procedure has to inscribe an image schema of the targeted mathematical notion; (5) The inquiry should be self-adaptive, not prescriptive, so that each child can gather the data they need when they need it; (6) The student should be able to move back and forth between embodied and symbolical control operations; and (7) The student should be supported in coordinating various meanings emerging from the activity by explicating relations among different strategies they discover.

Learning trajectories following this plan will likely be supported by, or fraught with interaction features exemplified by our study. Namely: (a) Students may encounter difficulty in homing on the core mathematical issues, if the EI technology is too obdurate, obtrusive, or otherwise unergonomic; (b) Students are liable to become competent at an EI task yet not know what they know, that is, they may perform well yet frame their performance by incompatible theory (Karmiloff-Smith, 1988), such that the pedagogical utility of their competence may remain untapped unless ushered forth; and (c) Students engaged in codependent EI tasks need to communicate effectively and work flexibly during their explorations, because they are liable to bear conflicting theories-in-action that, in turn, could impede the coproduction of targeted enactment objectives.

In sum, we view EI as bearing the capacity of supporting transformative teaching and learning. Specifically, EI enhances the implementation of visionary design frameworks, by which students should begin inquiry into complex mathematical concepts through presymbolic action-based reasoning (Forman, 1988; Nemirovsky, 2003; Thompson, 1993). More generally, we submit, EI activities constitute rewarding empirical contexts for research aiming to deepen and expand our field's understanding of how instructors discipline learners' perception of a shared domain of scrutiny (Stevens & Hall, 1998).

Whereas our work is in its early stages and our conclusions tentative, we hope to have conveyed enthusiasm over EI's unique instructional and theoretical affordances. Our future work will continue to seek improvements in both theory and design, availing of recent hands-free EI development. In this spirit, we offer the following lagniappe.

4.3 EI as a Transformative Learning Medium

New technological forms for enacting cultural practice bear the capacity to "restructurate" a domain so as to increase fluency, deepen understanding, and ultimately—as production prices plummet—democratize access (Blikstein & Wilensky, 2009; Brock & Price, 1980; Wilensky & Papert, 2010). In particular, even more so than "hands on" microworlds (e.g., Edwards, 1995; Kaput & West, 1994; Schwartz & Yerushalmy, 1993), EI-enabled "hands *in*" perceptuomotor exploration is ideally unencumbered by conceptually peripheral constraints of manipulation and typography inherent to traditional media and inscriptional systems. Thus, as the body settles into its skillful invisibility (Dreyfus, 1990), more cognitive resources should become available for tinkering creatively in the embodied microworld, by which the body instruments itself into a disciplinary domain of practice.

Yet restructurations of content may bear implications for restructuring theoretical models pertaining to this and other content learning. Specifically, whereas the hook-and-shift construct appears to agree with views of learning as discovery (Freudenthal, 1986) or creativity (Bamberger & Schön, 1983), our evidence of *un*mediated creative discovery complements sociocultural conceptualizations of learning as imitation, apprenticeship, or legitimate peripheral participation (Lave & Wenger, 1991; Lee & Majors, 2000; Rogoff, 1990). We thus anticipate that EI design stands to play a pivotal role in creating productive arenas for "dialectical" empirical research that strives to draw on both cognitive and sociocultural theory so as to build coherent models of conceptual ontogenesis in the interactive milieu (cf. diSessa, 2008).

Finally, the analytic orientation exemplified above suggests an intriguing line of research that we view as bearing great potential, namely conceptualizing how master mathematics teachers steer student engagement in activities centered on the mediated performance of physical actions. The ultimate goal of this line of research would be to develop an embodied-cognition model of grounded mathematics instruction. Yet this line of research could look to diverse cultural–historical forms of physical performance, such as music, dance, and the martial arts as research entries into pedagogical acumen. The skills inherent to these cultural practices might, at first blush, be viewed as aconceptual and, as such, hardly bearing on mathematical reasoning and learning. However, the recent theorization of mathematical cognition as grounded in multimodal, spatial–temporal imagistic simulations suggests that the phenomenology of mathematical reasoning draws on the same perceptuomotor resources as do human practices with manifest external physical production. From this perspective, mathematical concepts are articulated reifications of emergent embodied notions in conventional semiotic forms (see Radford, 2003) whose cultural mediation is enabled by shared biology and available artifacts (Núñez, Edwards, & Matos, 1999). Moreover, an implication of simulated multimodal action as the implicit fodder of mathematical cognition should encourage us to look closer at the pedagogical role of non-propositional information in educational exchanges (e.g., Alibali, Flevares, & Goldin-Meadow, 1997). There, and not in rote procedures for symbol manipulation, may lie the heart of the matter that students and researchers alike have been groping for.

References

Abrahamson, D. (2009). Embodied design: constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27-47.

Abrahamson, D., & Howison, M. (2010). *Embodied artifacts: coordinated action as an object-to-think-with. In D. L. Holton (Organizer & Chair) & J. P. Gee (Discussant), Embodied and enactive approaches to instruction: implications and innovations.* Paper presented at the annual meeting of the American Educational Research Association, April 30 - May 4.

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (under review). *Hooks and shifts: a dialectical study of mediated discovery.* (manuscript under review).

Alibali, M. W., Flevares, L. M., & Goldin-Meadow, S. (1997). Assessing knowledge conveyed in gesture: Do teachers have the upper hand? *Journal of Educational Psychology, 89*(1), 183-193.

Antle, A. N., Corness, G., & Droumeva, M. (2009). What the body knows: exploring the benefits of embodied metaphors in hybrid physical digital environments. In Ramduny-Ellis, D., Dix, A. J., Gill, S., & Hare, J. (Eds.), Physicality and interaction [Special issue]. *Interacting with Computers 21*(1&2), 66-75.

Artigue, M., Cerulli, M., Haspekian, M., & Maracci, M. (2009). Connecting and integrating theoretical frames: the TELMA contribution. In M. Artigue (Ed.), Connecting approaches to technology enhanced learning in mathematics: the TELMA experience [Special issue]. *International Journal of Computers for Mathematical Learning, 14*, 217-240.

Bamberger, J., & Schön, D. A. (1983). Learning as reflective conversation with

materials: notes from work in progress. *Art Education*, 66-73.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences, 22*, 577–660.

Barsalou, L. W. (2010). Grounded cognition: past, present, and future. *Topics in Cognitive Science, 2*(4), 716-724.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. D. English, M. G. Bartolini Bussi, G. A. Jones, R. Lesh & D. Tirosh (Eds.), *Handbook of international research in mathematics education, 2nd revised edition* (pp. 720-749). Mahwah, NG: Lawrence Erlbaum Associates.

Blikstein, P., & Wilensky, U. (2009). An atom is known by the company it keeps: constructing multi-agent models in engineering education. *International Journal of Computers for Mathematical Learning, 14*(2), 81-119.

Brock, W. H., & Price, M. H. (1980). Squared paper in the nineteenth century: Instrument of science and engineering, and symbol of reform in mathematical education. *Educational Studies in Mathematics, 11*(4), 365-381.

Brown, M. C., McNeil, N. M., & Glenberg, A. M. (2009). Using concreteness in education: real problems, potential solutions. *Child Development Perspectives, 3*, 160-164.

Carey, S., & Gelman, R. (Eds.). (1991). *The epigenesis of mind: essays on biology and cognition*. Hillsdale, NJ: Erlbaum Associates.

Church, R. B., & Goldin-Meadow, S. (1986). The mismatch between gesture and speech as an index of transitional knowledge. *Cognition, 23*, 43-71.

Cobb, P. (1989). A double-edged sword [Review of the book Intuition in science and mathematics]. *Journal for Research in Mathematics Education, 20*(2), 213-218.

Cress, U., Fischer, U., Moeller, K., Sauter, C., & Nuerk, H.-C. (2010). The use of a digital dance mat for training kindergarten children in a magnitude comparison task. In K. Gomez, L. Lyons & J. Radinsky (Eds.), *Learning in the Disciplines: Proceedings of the 9th International Conference of the Learning Sciences (ICLS 2010)* (Vol. 1 [Full Papers], pp. 105-112). International Society of the Learning Sciences: Chicago IL.

diSessa, A. A. (2008). A note from the editor. *Cognition and Instruction, 26*(4), 427-429.

diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *The Journal of the Learning Sciences, 13*(1), 77-103.

Dourish, P. (2001). *Where the action is: The foundations of embodied interaction*. Cambridge, MA: M.I.T. Press.

Dreyfus, H. L., & Dreyfus, S. E. (1999). The challenge of Merleau-Ponty's phenomenology of embodiment for cognitive science. In G. Weiss & H. F. Haber (Eds.), *Perspectives on embodiment: The intersections of nature and culture* (pp. 103-120). New York: Routledge.

Edwards, L. (1995). Microworlds as representations. In A. A. diSessa, C. Hoyles & R. Noss (Eds.), *Computers and exploratory learning* (pp. 127-154). New York: Springer.

Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the construction of mathematical meaning [Special Issue]. *Educational Studies in Mathematics, 70*(2).

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettenen & R.-L. Punamäki` (Eds.), *Perspectives on activity theory* (pp. 19-38). Cambridge, UK: Cambridge University Press.

Forman, G. (1988). Making intuitive knowledge explicit through future technology. In G. Forman & P. B. Pufall (Eds.), *Constructivism in the computer age*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing Company.

Goldin, G. A. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65). Hillsdale, NJ: Lawrence Erlbaum Associates.

Goldin-Meadow, S., & Beilock, S. L. (2010). Action's influence on thought: the case of gesture. *Perspectives on Psychological Science, 5*(6), 664-674.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 603-633.

Hall, R. (1996). Representation as shared activity: situated cognition and Dewey's cartography of experience. *Journal of the Learning Sciences, 5*(3), 209-238.

Hall, R. (2001). Cultural artifacts, self regulation, and learning: Commentary on Neuman's "Can the Baron von Munchhausen phenomenon be solved?" *Mind, Culture & Activity, 8*(1), 98-108.

Hoffmann, M. H. G. (2003). Peirce's 'diagrammatic reasoning' as a solution of the learning paradox. In G. Debrock (Ed.), *Process pragmatism: Essays on a quiet philosophical revolution* (pp. 121-143). Amsterdam: Rodopi.

Hommel, B., Müsseler, J., Aschersleben, G., & Prinz, W. (2001). The theory of event coding (TEC): A framework for perception and action planning. *Behavioral and Brain Sciences, 24*, 849-878.

Hornecker, E., & Dünser, A. (2009). Of pages and paddles: children's expectations and mistaken interactions with physical-digital tools. *Interacting with Computers, 21*, 95-107.

Howison, M., Trninic, D., Reinholz, D., & Abrahamson, D. (in press). The Mathematical Imagery Trainer: from embodied interaction to conceptual learning. In G. Fitzpatrick & C. Gutwin (Eds.), *Proceedings of the ACM CHI Conference on Human Factors in Computing Systems (CHI 2011), Vancouver, May 7-12, 2011*.

Husserl, E. (1970). *The crisis of European sciences and transcendental phenomenology* (D. Carr, Trans.). Evanston, IL: Northwestern University Press. (Original work 1936).

Johnson, M. L. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason*. Chicago: Chicago University Press.

Kamii, C. K., & DeClark, G. (1985). *Young children reinvent arithmetic: implications of Piaget's theory*. New York: Teachers College Press.

Kammer, R. (1986). *Way of the Sword: Tengu-Geijutsu-Ron of Chozan Shissai*. London: Arkana.

Kaput, J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp.

237-287). Albany, NY: SUNY.

Karmiloff-Smith, A. (1988). The child is a theoretician, not an inductivist. *Mind & Language, 3*(3), 183-195.

Karmiloff-Smith, A. (1992). *Beyond modularity: a developmental perspective on cognitive science*. Cambridge, MA: MIT Press.

Karmiloff-Smith, A., & Inhelder, B. (1978). If you want to get ahead, get a theory. In P. N. Johnson-Laird & P. C. Wason (Eds.), *Thinking. Readings in cognitive science* (pp. 293-306). Cambridge, MA: Cambridge University Press.

Kirsch, D. (2006). Distributed cognition: a methodological note. In Harnad, S., & Dror, I. E. (Eds.), Distributed cognition [Special issue]. *Pragmatics & Cognition, 14*(2), 249-262.

Klemmer, S. R., Hartmann, B., & Takayama, L. (2006). How bodies matter: five themes for interaction design. In *Proceedings of the 6th conference on Designing Interactive Systems*: ACM.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte: Information Age Publishing.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.

Lee, C., & Majors, Y. (2000). *Cultural modeling's response to Rogoff's challenge: Understanding apprenticeship, guided participation and participatory appropriation in a culturally responsive, subject matter specific context.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Marshall, P., Cheng, P. C.-H., & Luckin, R. (2010). Tangibles in the balance: a discovery learning task with physical or graphical materials. In *Proceedings of the 4th ACM International Conference on Tangible, Embedded, and Embodied Interaction* (pp. 153-160). NYC: ACM.

McLuhan, M. (1964). *Understanding media: The extensions of man*. New York: The New American Library.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In R. Nemirovsky & M. Borba (Coordinators), Perceptuo-motor activity and imagination in mathematics learning (Research Forum). In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Twenty Seventh Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105-109). Honolulu, Hawaii: Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.

Neuman, Y. (2001). Can the Baron von Münchausen phenomenon be solved? An activity-oriented solution to the learning paradox. *Mind, Culture & Activity, 8*(1), 78-89.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: working for*

cognitive change in school. New York: Cambridge University Press.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: connecting the visual with the symbolic. *Educational Studies in Mathematics, 33*(2), 203-233.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*, 45-65.

Olive, J. (2000). Computer tools for interactive mathematical activity in the elementary school. *International Journal of Computers for Mathematical Learning, 5*(3), 241-262.

Papert, S. (1980). *Mindstorms: children, computers, and powerful ideas*. NY: Basic Books.

Piaget, J. (1968). *Genetic epistemology* (E. Duckworth, Trans.). New York: Columbia University Press.

Piaget, J., & Inhelder, B. (1969). *The psychology of the child* (H. Weaver, Trans.). NY: Basic Books (Original work published 1966).

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205-235). Rotterdam: Sense Publishers.

Pufall, P. (1988). Function in Piaget's system: some notes for constructors of microworlds. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37-70.

Raymaekers, C. (Ed.) (2009). Editorial on enactive interfaces [Special issue]. *Interacting with Computers, 21*((1&2)).

Reinholz, D., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: embodied artifacts and the guided emergence of mathematical meaning. In P. Brosnan, D. Erchick & L. Flevares (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 32)* (Vol. VI, Ch. 18: Technology, pp. 1488 – 1496). Columbus, OH: PME-NA.

Resnick, L. B. (1992). From protoquantities to operators: building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-429). Hillsdale: LEA.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. NY: Oxford University Press.

Roth, W.-M. (2009). Mathematical representation at the interface of body and culture. In B. Sriraman & L. D. English (Eds.), *International perspectives on mathematics education -- cognition, equity, and society*. Charlotte, NC: Information Age

Publishing.

Saxe, G. B. (2004). Practices of quantification from a sociocultural perspective. In K. A. Demetriou & A. Raftopoulos (Eds.), *Developmental change: theories, models, and measurement* (pp. 241-263). NY: Cambridge University Press.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*, 253-286.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1991). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.

Schön, D. A. (1992). Designing as reflective conversation with the materials of a design situation. *Research in Engineering Design, 3*, 131-147.

Schunn, C. (2008). Engineering educational design. *Educational Designer, 1*(1).

Schwartz, J. L., & Yerushalmy, M. (1993). *The Geometric Super Supposer: A laboratory for mathematical exploration, conjecture and invention*. Pleasantville, N.Y.: Sunburst Communications.

Sebanz, N., & Knoblich, G. (2009). Prediction in joint action: what, when, and where. *Topics in Cognitive Science, 1*(2), 353-367.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences, 11*(2&3), 319-357.

Sloutsky, V. M., Kaminski, J. A., & Heckler, A. F. (2005). The advantage of simple symbols for learning and transfer. *Psychonomic Bulletin and Review, 12*(3), 508-513.

Stevens, R., & Hall, R. (1998). Disciplined perception: learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107-149). New York: Cambridge University Press.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 3*, 165-208.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-236). Albany, NY: SUNY.

Trninic, D., Reinholz, D., Howison, M., & Abrahamson, D. (2010). Design as an object-to-think-with: semiotic potential emerges through collaborative reflective conversation with material. In P. Brosnan, D. Erchick & L. Flevares (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 32)* (Vol. VI, Ch. 18: Technology, pp. 1523 – 1530). Columbus, OH: PME-NA.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: cognitive science and human experience*. Cambridge, MA: M.I.T. Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127–174). New York: Academic Press.

Vergnaud, G. (2009). The theory of conceptual fields. In T. Nunes (Ed.), Giving meaning to mathematical signs: psychological, pedagogical and cultural processes. *Human Development [Special Issue], 52*, 83-94.

Vygotsky, L. S. (1930/1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Wertsch, J. V. (1979). From social interaction to higher psychological processes: a clarification and application of Vygotsky's theory. *Human Development, 22*(1), 1-22.

White, T. (2008). Debugging an artifact, instrumenting a bug: Dialectics of instrumentation and design in technology-rich learning environments. *International Journal of Computers for Mathematical Learning, 13*(1), 1-26.

Wilensky, U., & Papert, S. (2010). Restructurations: reformulations of knowledge disciplines through new representational forms. In J. Clayson & I. Kallas (Eds.), *Proceedings of the Constructionism 2010 Conference*. Paris.

Yap, D.-F., So, W.-C., Melvin Yap, J.-M., Tan, Y.-Q., & Teoh, R.-L. S. (2011). Iconic gestures prime words. *Cognitive Science, 35*(1), 171-183.